

2019

MATHEMATICS

( Major )

Paper : 3.2

( Linear Algebra and Vector )

Full Marks : 80

Time : 3 hours

*The figures in the margin indicate full marks  
for the questions*

GROUP—A

( Linear Algebra )

( Marks : 40 )

1. Answer the following as directed :

1×7=7

(a) If  $V = R^n(R)$ , then

$$W = \{(v_1, v_2, \dots, v_n) : v_1 + v_2 + \dots + v_n = 1\}$$

is a subspace of  $V$ .

( Disprove it )



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- (b) If  $v_i = v_j$  for some  $i \neq j$ , then the sequence  $v_1, v_2, \dots, v_n$  of vectors in a vector space is

(i) linearly independent

(ii) linearly dependent

( Choose the correct option )

- (c) Consider the complex field  $C$  which contains the real field  $R$ . Show that  $\{1, i\}$  is a basis of the vector space  $C$  over  $R$ , where  $i = \sqrt{-1}$ .

- (d) Suppose  $T: R^5 \rightarrow R^2$  is a linear transformation defined by  $T(x) = Ax$  for some matrix  $A$  and for each  $x$  in  $R^5$ . How many rows and columns does  $A$  have?

- (e) For a linear operator (matrix)  $T$ , the scalar 0 is an eigenvalue of  $T$  if and only if  $T$  is singular.

( Write True or False )

- (f) Find the minimal polynomial  $m(t)$  of the following matrix :

$$M = \begin{bmatrix} -5 & 4 \\ 2 & 9 \end{bmatrix}$$

( 3 )

- (g) If  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ , then  $x \mapsto Ax$  is a linear transformation on  $C^2$ , where  $C$  is the complex field. Show that  $v = \begin{bmatrix} 1 \\ -i \end{bmatrix}$  is an eigenvector of  $A$ .

2. Answer the following questions :

2×4=8

- (a) Determine if the columns of the matrix

$$A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$$

are linearly independent.

- (b) Let  $M_{2 \times 2}$  be the vector space of all  $2 \times 2$  matrices, and define  $T: M_{2 \times 2} \rightarrow M_{2 \times 2}$

by  $T(A) = A + A^T$ , where  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

Show that  $T$  is a linear transformation.

- (c) Let  $a_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ ,  $a_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ ,  $b_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$ ,

$b_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ , and consider the bases for  $R^2$

given by  $E = \{a_1, a_2\}$  and  $F = \{b_1, b_2\}$ . Find the change-of-coordinate matrix from  $E$  to  $F$ .



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- (d) If  $\lambda$  is an eigenvalue of a linear operator  $T: V \rightarrow V$ , then prove that the set  $E_\lambda$  of all eigenvectors belonging to  $\lambda$  is a subspace of  $V$ .

3. Answer any one part :

5

- (a) Let  $V_1$  and  $V_2$  be vector spaces over the same field  $F$ . For any linear transformation  $T: V_1 \rightarrow V_2$ , prove that  $r(T) \leq \min(\dim V_1, \dim V_2)$ , where  $r(T)$  denotes the rank of  $T$ .
- (b) Define the dual space  $V^*$  of a vector space  $V$  and prove that if  $V$  is the finite-dimensional vector space over a field  $F$ , then for any  $u (\neq 0)$  in  $V$  there exists  $g \in V^*$  such that  $g(u) \neq 0$ .

4. Answer the following questions :

- (a) If  $u_1, u_2, \dots, u_n$  are non-zero linearly dependent vectors in a vector space  $V$  over a field  $F$ , then prove that for some  $i$ ,  $2 \leq i \leq n$ ,  $u_i$  is a linear combination of its predecessors  $u_1, u_2, \dots, u_{i-1}$  and the subspace spanned by  $\{u_1, u_2, \dots, u_n\}$  is same as that spanned by  $\{u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_n\}$ .

10

( 5 )

Or

Suppose  $V$  is a finite dimensional vector space over a field  $F$  and  $U$  is a subspace of  $V$ . Prove that there is a subspace  $W$  of  $V$  such that  $V = U \oplus W$ .

10

- (b) (i) If a linear transformation  $T: V \rightarrow V$  has  $n$  distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ , then prove that  $V$  has an ordered basis  $\{u_1, u_2, \dots, u_n\}$  such that the matrix of  $T$  related to this basis is

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

- (ii) State the existence and uniqueness theorem of solution of a system of linear equations. Determine the existence and uniqueness of the solution of the system whose augmented matrix after row reduced is

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

5+5=10



( 6 )

Or

What do we mean by the minimal polynomial of a matrix (linear operator)?  
Let  $T$  be the operator on  $R^2$  which projects each vector onto the  $x$ -axis, parallel to the  $y$ -axis :

$$T(x, y) = (x, 0)$$

Show that  $T$  is linear. What is the minimal polynomial for  $T$ ? 10

GROUP—B

( Vector )

( Marks : 40 )

5. Answer the following questions : 1×3=3

(a) Prove that  $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ .

(b) If  $\vec{a}, \vec{b}, \vec{c}$  are non-coplanar vectors, then define the reciprocal vector of  $\vec{a}$ .

(c) If the four vectors  $\vec{a}, \vec{b}, \vec{c}, \vec{d}$  are coplanar, then show that  
 $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$

6. Find the constant  $\lambda$  such that the following vectors are coplanar : 2

$$\vec{a} = 2\vec{i} - \vec{j} + \vec{k}, \quad \vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}, \\ \vec{c} = 3\vec{i} + \lambda\vec{j} + 5\vec{k}$$

( 7 )

7. Answer the following questions :

(a) Prove that for any three vectors  $\vec{a}, \vec{b}$ , and  $\vec{c}$ ,  $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ . 5

(b) Prove that the necessary and sufficient condition for a vector  $\vec{v}(t)$  to have a constant direction is

$$\vec{v} \times \frac{d\vec{v}}{dt} = \vec{0} \quad 5$$

(c) Prove that

$$\text{curl}(\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} - \vec{B} \text{ div } \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A} \text{ div } \vec{B} \quad 5$$

Or

Taking  $\vec{f} = x^2 y \vec{i} + xz \vec{j} + 2yz \vec{k}$ , verify that  
 $\text{div}(\text{curl } \vec{f}) = 0$ . 5

8. Answer the following questions :

(a) (i) If  $\vec{r} = a \cos t \vec{i} + a \sin t \vec{j} + at \tan \alpha \vec{k}$ , then find

$$\left| \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \right| \text{ and } \left( \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \right) \cdot \frac{d^3\vec{r}}{dt^3} \quad 2+3=5$$

(ii) A particle moves along the curve  $x = t^3 + 1, y = t^2, z = 2t + 5$ , where  $t$  is the time. Find the components of its velocity and acceleration at  $t = 1$  in the direction  $\vec{i} + \vec{j} + 3\vec{k}$ . 5



Or

(i) If  $\vec{a}$ ,  $\vec{b}$ ,  $\vec{c}$  be three unit vectors such that  $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$ , then find the

angles which  $\vec{a}$  makes with  $\vec{b}$  and  $\vec{c}$ , where  $\vec{b}$  and  $\vec{c}$  being the parallel. 4

(ii) If  $\frac{d\vec{u}}{dt} = \vec{w} \times \vec{u}$ ,  $\frac{d\vec{v}}{dt} = \vec{w} \times \vec{v}$ , then show

that  $\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{w} \times (\vec{u} \times \vec{v})$ . 3

(iii) Prove that  $\text{div } \vec{r} = 3$ . 3

(b) If  $\vec{F} = y\vec{i} - x\vec{j}$ , then evaluate  $\int_C \vec{F} \cdot d\vec{r}$  from

(0, 0) to (1, 1) along the following paths C:

(i) The parabola  $y = x^2$

(ii) The straight lines from (0, 0) to (1, 0) and then to (1, 1)

(iii) The straight line joining (0, 0) and (1, 1). 4+3+3=10

Or

If  $\vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}$  and S is the surface of the sphere

$$x^2 + y^2 + z^2 = a^2, \quad 0 \leq x, y, z \leq a$$

then evaluate  $\iint_S \vec{F} \cdot \vec{n} dS$ , where  $\vec{n}$  is a unit vector along the outward down normal to the sphere S. 10

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