Remarks: It should be remembered that P.I. appears due to X in (1). Hence if a linear differential equation with constant coefficients is given with X = 0, then its general solution will not involve P.I. and so for such differential equations, the general solution will be given by y = C.F.

Auxiliary equation:

Consider the differential equation (1) with X = 0 i.e.

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = 0 \quad \text{or} \quad f(D)\, y = 0 \quad \text{--- (A)}$$

Assume that $y = e^{mx}$ is a solution of this equation, then since $y = e^{mx}$

$$\therefore Dy = m e^{mx}$$

$$D^2 y = m^2 e^{mx}$$

$$\vdots$$

$$D^n y = m^n e^{mx}$$

$\therefore$ (A) $\Rightarrow$

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n) e^{mx} = 0$$

which will hold if

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0$$

$$\Rightarrow f(m) = 0 \quad \text{--- (B)}$$

This equation is called the auxiliary equation. On comparing (A) and (B), we see that the auxiliary equation $f(m) = 0$ will give the same values of m as the equation $f(D) = 0$ gives of D.

Working rule for finding complementary function:

Case (I):

First suppose that the auxiliary equation has n distinct roots $m_1, m_2, \ldots, m_n$ then the c.F. is given by $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Again, if the auxiliary equation has the real root m_k occurring k times, and if, further the remaining roots of the auxiliary equation are distinct real numbers $m_{k+1}, m_{k+2}, \ldots, m_K$, then c.F. is given by

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{m_k x} + c_{k+1} e^{m_{k+1} x} + \cdots + C_m e^{m_n K}$$

Example: let us solve $(D^2 - 3D + 2)y = 0$

Here the auxiliary eq^n is $m^2 - 3m + 1 = 0$

$$\Rightarrow m = 1, 2$$

Hence c.F. is $c_1 e^x + c_2 e^{2x}$

Since here $X = 0$, so the general solution is

$$y = C.F. = c_1 e^x + c_2 e^{2x}$$

Consider another example: $(D^3 - 3D + 2)y = 0$

so that the auxiliary equation is $D^3 - 3D + 2 = 0$

$$\Rightarrow (D-1)(D-1)(D+2) = 0$$

which gives $D = 1, 1, -2$. Since 1 occurs two times, the c.F. and so the general solution of the given equation is

$$y = (c_1 + c_2 x)e^x + c_3 e^{-2x}$$ where c_1, c_2 and c_3 are arbitrary constants

Case(ii) Let $\alpha \pm i\beta$ be a pair of complex roots. Then the corresponding part of the C.F. may be written in one of the following forms

$$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x),\ c_1 e^{\alpha x} \cos(\beta x + c_2)$$

or $c_1 e^{\alpha x} \sin(\beta x + c_2)$

If, however, the auxiliary equation has two equal pairs of complex roots, $\alpha + i\beta$ and $\alpha - i\beta$ say occur twice, the corresponding part of the C.F. is written as

$$e^{\alpha x}\{(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x) \sin \beta x\}$$

Case(iii): If a pair of the roots of the auxiliary equation involves surds, say $\alpha \pm \sqrt{\beta}$, where β is positive, then the corresponding part of C.F. may be written in one of the following three forms

$$e^{\alpha x}[c_1 \cosh(x\sqrt{\beta}) + c_2 \sinh(x\sqrt{\beta})],\ c_1 e^{\alpha x} \cosh(x\sqrt{\beta} + c_2),$$

$$c_1 e^{\alpha x} \sinh(x\sqrt{\beta} + c_2)$$

For example: Solve $(D^2 + 6D + 4)y = 0$

Here auxiliary eqn is $D^2 + 6D + 4 = 0$

$$\Rightarrow D = -3 \pm \sqrt{5}$$

Hence the general solution is

$$y = e^{-3x}(c_1 \cosh x\sqrt{5} + c_2 \sinh x\sqrt{5})$$