Theorem: General solutions of Homogeneous Equations

Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

with p and q continuous on the open interval I.

If Y is any solution whatsoever of eqn $y'' + p(x)y' + q(x)y = 0$ on I, then there exist numbers c_1 and c_2 such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) \text{ for all } x \text{ in } I.$$

Ex) Show that e^{2x} and e^{3x} are linearly independent solution of $y'' - 5y' + 6y = 0$ and find the solution $y(x)$ with property $y(0) = 0$ and $y'(0) = 1$

Soln: Given equation is

$$y'' - 5y' + 6y = 0 \quad \text{---(1)}$$

Let $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$ ---(2)

$\therefore$ $y_1'(x) = 2e^{2x}$ and $y_2'(x) = 3e^{3x}$

and $y_1''(x) = 4e^{2x}$, $y_2''(x) = 9e^{3x}$

Then $y_1''(x) - 5y_1'(x) + 6y_1(x) = 4e^{2x} - 10e^{2x} + 6e^{2x} = 0$

and $y_2''(x) - 5y_2'(x) + 6y_2(x) = 9e^{3x} - 15e^{3x} + 6e^{3x} = 0$

Now, Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$$

$$= 3e^{5x} - 2e^{5x}$$

$$= e^{5x} \neq 0$$

$\therefore$ $W(x) \neq 0$

showing that e^{2x} and e^{3x} are linearly independent, therefore $y_1(x)$ and $y_2(x)$ are solutions of (1).

The general solution of (1) is given by

$$y(x) = c_1 e^{2x} + c_2 e^{3x} \quad --- (3)$$

$$\therefore \quad y'(x) = 2c_1 e^{2x} + 3c_2 e^{3x} \quad --- (4)$$

using the given Property in (3) and (4) we get

$$0 = c_1 + c_2 \quad \text{and} \quad 1 = 2c_1 + 3c_2$$

$$\Rightarrow 2c_1 - 3c_1 = 1$$

$$\Rightarrow c_1 = -1$$

$$\therefore c_2 = 1$$

Hence (3) $\Rightarrow$ $y(x) = -e^{2x} + e^{3x}$ is required sol^n.

Ex) Show that $e^x \sin x$ and $e^x \cos x$ are linearly independent solutions of $y'' - 2y' + 2y = 0$ and find the general solution $y(x)$ with property $y(0) = 2$ and $y'(0) = -3$

Ex) Show that $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ are linearly independent solution of $y'' + y = 0$. Determine the constants c_1 and c_2 s.that solution $\sin x + 3\cos x = c_1 y_1(x) + c_2 y_2(x)$.

Ex) Show that the solution e^x, e^{-x} and e^{2x} of $y'''(x) - y''(x) - y'(x) + 2y = 0$ are linearly independent.

Sol^n: Given eq^n is

$$y'''(x) - y''(x) - y'(x) + 2y = 0 \quad ---(1)$$

Let $y_1(x) = e^x$, $y_2(x) = e^{-x}$ and $y_3(x) = e^{2x}$ --(2)

Then $y_1'(x) = e^x$, $y_1''(x) = e^{x}$, $y_1'''(x) = e^x$

$\therefore$ $y_1'''(x) - 2y_1''(x) - y_1'(x) + 2y_1(x) = e^x - 2e^x - e^x + 2e^x$

$= 0$

Hence $y_1(x) = e^x$ is the solution of (1).

Similarly we can show that e^{-x} and e^{2x} are also solutions of (1).

Now, Wronskian $W(x)$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix}$$

$$= e^x e^{-x} e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix}$$

$$= e^{2x}(-6)$$

$$= -6e^{2x} \neq 0 \qquad \text{i.e. } W(x) \neq 0$$

Hence y_1, y_2 and y_3 are linearly independent solutions of (1).