* A set of vectors $x_1, x_2, \ldots x_n$ is said to be linearly dependent if there exist n scalars (numbers) $k_1, k_2, \ldots k_n$, not all zero s.t

$$k_1 x_1 + k_2 x_2 + \cdots + k_n x_n = 0$$

where '0' denotes the vector whose components are all zero.

A set of vectors $x_1, x_2, \ldots x_n$ is said to be linearly independent if every relation of the type

$$k_1 x_1 + k_2 x_2 + k_3 x_3 + \ldots + k_n x_n = 0$$

implies $k_1 = k_2 = k_3 = \ldots = k_n = 0$

Echelon Matrix:

(1) All zero rows, if any, are at the bottom of the matrix.

Rank = Number of non-zero row in the echelon form of the matrix.

* Elementary transformations do not change the rank of a matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 0 & 1 & 1 & 3 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank = 2

Ex) Show that the eqns

$$x+y+z=-3$$

$$3x+y-2z=-2$$

$$2x+4y+7z=7$$

are not consistent.

Sol^{n} The given system of eqn's is equivalent to the single matrix eqn

$$AX = B$$

Where $A=\begin{bmatrix}1 & 1 & 1\\ 3 & 1 & -2\\ 2 & 4 & 7\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$, $B=\begin{bmatrix}-3\\ -2\\ 7\end{bmatrix}$

The Augmented matrix is

$$C=[A:B]=\left[\begin{array}{ccc:c}1 & 1 & 1 & -3\\ 3 & 1 & -2 & -2\\ 2 & 4 & 7 & 7\end{array}\right]$$

We shall reduce the augmented matrix, C to Echelon form by applying elementary row transformations only.

$$C = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

above matrix is the echelon form of the matrix C. We have rank of C = the number of non-zero rows in this echelon form = 3.

Also, by same E-row transformations, we get,

$$A \sim \left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{array}\right]$$, obviously rank of A = 2

Since Rank of A ≠ Rank of C, therefore the given eq^n's are inconsistent.

i.e they have no solution.

Ex) Show that the eq^n's

$x+y+z=6$

$x+2y+3z=14$

$x+4y+7z=30$

are consistent and solve them.

Sol^n The given system of eq^n's is equivalent to the single matrix eqn

$$AX=B$$

where $A=\begin{bmatrix}1&1&1\\1&2&3\\1&4&7\end{bmatrix}$, $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$, $B=\begin{bmatrix}6\\14\\30\end{bmatrix}$

The augmented matrix is

$$C=[A:B]=\left[\begin{array}{ccc:c}1&1&1&6\\1&2&3&14\\1&4&7&30\end{array}\right]\begin{matrix}\\R_2\to R_2-R_1\\R_3\to R_3-R_1\end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right] \quad R_3 \to R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

~~The~~ Above matrix is the echelon form of the matrix C. We have rank of C = 2.

By the same elementary tranformations, we get,

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Obviously rank of A = 2.

$\because$ Rank of A = Rank of C,

$\therefore$ the given eqns are consistent.

~~Here / rank of A = rank of C & the~~

Here the number of unknowns is 3.

Since rank of A is less than the number of unknowns, therefore the given system will have an infinite number of solutions.

The given system of eq's is equivalent to the matrix eqn

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

This matrix eqn is equivalent to the system of eq's

$$x + y + z = 6$$
$$y + 2z = 8$$

$\therefore$ $y = 8 - 2z$, $x = 6 - y - z$

$= 6 - 8 + 2z - z$

$= z - 2$

taking $z = c$, we see that $x = c - 2$, $y = 8 - 2c$, $z = c$ constitute the general soln of the given system, where c is an arbitrary constant.

Ex) Show that the eqns

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$

are consistent and solve them.

Sol^n The given system of eqns is equivalent to the single matrix eqn

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

The augmented matrix

$$C = [A : B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

$\therefore$

Performing $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$, we get:

$$C \sim \left[\begin{array}{cc|cc} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right] \quad R_2 \to R_2 - R_3$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - 6R_2 \\ R_4 \to R_4 - 3R_2 \end{array}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 2 & 8 \end{array}\right] \quad \begin{array}{l} R_3 \to \frac{1}{5} R_3 \\ R_4 \to \frac{1}{2} R_4 \end{array}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right] \quad R_4 \to R_4 - R_3$$

$$\sim \left[\begin{array}{cc|cc} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the matrix C has been reduced to echelon form, we have rank C = 3.

Also,

$$A \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have rank A = 3.

∴ rank C = rank A, therefore the given eqns are consistent. Since rank A = 3 = the number of unknowns, therefore the given eqns have unique solution. The given eqns are equivalent to the eqns

$$x + 2y - z = 3$$
$$-y = -4,$$
$$z = 4$$

Thus give $z = 4$, $y = 4$, $x = -1$.

Ex) Solve the system of eqn's

$$2x - 2y + 5z + 3w = 0$$

$$4x - y + z + w = 0$$

$$3x - 2y + 3z + 4w = 0$$

$$x - 3y + 7z + 6w = 0$$

Soln

$$AX = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

$$A = \begin{bmatrix} 2 & -2 & 5 & 3 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 1 & -3 & 7 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 4 & -1 & 1 & 1 \\ 3 & -2 & 3 & 4 \\ 2 & -2 & 5 & 3 \end{bmatrix} \text{ by } R_1 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 11 & -27 & -23 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix}$$ by $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 7 & -18 & -14 \\ 0 & 4 & -9 & -9 \end{bmatrix}$$ by $R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 28 & -72 & -56 \\ 0 & 4 & -9 & -9 \end{bmatrix}$$ by $R_3 \to 4R_3$

$$\sim \begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 0 & -9 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ by $R_3 \to R_3 - 7R_2$, $R_4 \to R_4 - R_2$

Since the rank of A is less than the number of unknowns, therefore the given system of eq^{n}s possesses non-zero sol^{n}s.

The given system of eqns will have

$4-3=1$ linearly independent soln.

We shall assign arbitrary value to

$n-r = 4-3 = 1$ variable and the remaining variables shall be found in terms of these. The given system of eqns is equivalent to

$$\begin{bmatrix} 1 & -3 & 7 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 0 & -9 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x - 3y + 7z + 6w = 0$$

$$4y - 9z - 9w = 0$$

$$-9z + 7w = 0$$

From these, we get,

$$z = \frac{7}{9}w, \quad y = 4w, \quad x = \frac{5}{9}w$$

taking $w = c$, we get,

$$z = \frac{7}{9}c, \quad w = c, \quad x = \frac{5}{9}c, \quad y = 4c,$$

constitute the general

s.l^{r} of the given system.

* The number of linearly independent sol^{n}s of m homogeneous linear eq^{n}s in n variables, $AX=0$, is $(n-r)$, where r is the rank of the matrix A.