Ex) Show that $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$ are two solution of the equation $y'' - 9y = 0$. Also verify the principle of super position.

Sol^n:- Given equation is

$$y'' - 9y = 0 \quad \text{-----}(1)$$

Now let $y(x) = e^{3x}$

$\therefore$ $y'(x) = 3e^{3x}$

$y''(x) = 9e^{3x}$

Now putting these values in (1)

$$y'' - 9y = 9e^{3x} - 9e^{3x} = 0$$

Hence e^{3x} is the solution of (1).

Again let $y = e^{-3x}$

$\therefore$ $y' = -3e^{-3x}$ and $y'' = 9e^{-3x}$

Now, putting these value in (1),

$$y'' - 9y = 9e^{-3x} - 9e^{-3x} = 0$$

Hence e^{-3x} is the solution of (1).

$\therefore$ $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$ are two solutions of the differential equation (1).

Now, we have to verify that $y = c_1 e^{3x} + c_2 e^{-3x}$ is also the solution of (1).

So

$$y'' - 9y$$

$$= 9c_1 e^{3x} + 9c_2 e^{-3x} - 9c_1 e^{3x} - 9c_2 e^{-3x}$$

$$= 0$$

Thus $y = c_1 e^{3x} + c_2 e^{-3x}$ is general sol^n of $y'' - 9y = 0$.

Ex) Show that $y_1 = x^2$ and $y_2 = 1$ are two solutions of the homogeneous non linear ordinary differential equation $y''y - xy' = 0$. Also verify the principle of super position.

Ex) Show that $y_1(x) = e^{-5x}$ and $y_2(x) = e^{2x}$ are the solutions of $y'' + 3y' - 10y = 0$ and also verify the principle of super position at initial condition $y(0) = 4$ and $y'(0) = -2$

The Wronskian:

Let $y_1(x)$ and $y_2(x)$ be two solutions of the second order differential equation

$$a_0(x)\, y''(x) + a_1(x)\, y'(x) + a_2(x)\, y(x) = 0$$

Then the Wronskian of $y_1(x)$ and $y_2(x)$ is denoted and defined as the determinant

$$W(y_1, y_2) \text{ or } W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

If the Wronskian of the function $y_1(x)$ and $y_2(x)$ is zero then the functions $y_1(x)$ and $y_2(x)$ are called linearly dependent.

If the Wronskian of the function $y_1(x)$ and $y_2(x)$ is non-zero then the functions $y_1(x)$ and $y_2(x)$ are called linearly independent.

Ex) If $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$ are two solutions of differential equation $y'' + 9y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions.

Sol^n: The Wronskian of $y_1(x)$ and $y_2(x)$

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix} = -3\sin^2 3x - 3\cos^2 3x$$

$$= -3(1)$$

$$= -3 \neq 0$$

Since $W(x) \neq 0$, $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $y'' + 9y = 0$.

Ex) Prove that $\sin 2x$ and $\cos 2x$ are solutions of the differential equation $y'' + 4y = 0$ and these solutions are linearly independent.