## Unit - 2

### Finite differences:

Suppose we have a function $y = f(x)$, where $x$ can take the values $a, a+h, a+2h, \ldots, a+nh$. Let the corresponding values of $y$ i.e. $f(x)$ be $f(a), f(a+h), f(a+2h), \ldots, f(a+nh)$. The expression $f(a+h) - f(a)$ is called the first difference of $f(a)$ and we denote this difference by $\Delta f(a)$.

Thus $\Delta f(a) = f(a+h) - f(a)$

Similarly $f(a+2h) - f(a+h), \ldots, f(a+nh) - f\{a+(n-1)h\}$ are all called the first differences and generally denoted as

$$\Delta f(x) = f(x+h) - f(x), \quad x = a, a+h, \ldots, a+(n-1)h. \quad \ldots (1)$$

Here $\Delta$ is an operator and is called a forward difference operator, $h$ is called the interval of differencing.

If we operate $\Delta$ on (1), we get second difference of the function values.

$$\Delta\{\Delta f(x)\} = \Delta\{f(x+h) - f(x)\}$$

$$\Rightarrow \Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$$

$$\Rightarrow \Delta^2 f(x) = \{f(x+2h) - f(x+h)\} - \{f(x+h) - f(x)\}$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

Here $\Delta^2$ represents that the operation of difference has been done twice.

We may continue to define third difference as

$$\Delta^3 f(x) = \Delta\{\Delta^2 f(x)\} = \Delta\{f(x+2h) - 2f(x+h) + f(x)\}$$

$$= \Delta f(x+2h) - 2\Delta f(x+h) + \Delta f(x)$$

$$= \{f(x+3h) - f(x+2h)\} - 2\{f(x+2h) - f(x+h)\} +$$

$\{f(x+h) - f(x)\}$

$\Delta^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$

and so on.

In general the nth forward difference is given by

$$\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$$

* Difference table:

| Argument $x$ | Entry $f(x)$ | First differences $\Delta f(x)$ | Second differences $\Delta^2 f(x)$ |
|---|---|---|---|
| $a$ | $f(a)$ | | |
| | | $f(a+h) - f(a) = \Delta f(a)$ | |
| $a+h$ | $f(a+h)$ | | $\Delta f(a+h) - \Delta f(a) = \Delta^2 f(a)$ |
| | | $f(a+2h) - f(a+h) = \Delta f(a+h)$ | |
| $a+2h$ | $f(a+2h)$ | | $\Delta f(a+2h) - \Delta f(a+h) = \Delta^2 f(a+h)$ |
| | | $f(a+3h) - f(a+2h) = \Delta f(a+2h)$ | |
| $a+3h$ | $f(a+3h)$ | | $\Delta f(a+3h) - \Delta f(a+2h) = \Delta^2 f(a+2h)$ |
| | | $f(a+4h) - f(a+3h) = \Delta f(a+3h)$ | |
| $a+4h$ | $f(a+4h)$ | | |
| ⋮ | ⋮ | ⋮ | ⋮ |

Ex) Given $f(0)=3$, $f(1)=12$, $f(2)=81$, $f(3)=200$, $f(4)=100$ and $f(5)=8$. Form a difference table and find $\Delta^5 f(0)$.

Solⁿ:

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ | $\Delta^5 f(x)$ |
|---|---|---|---|---|---|---|
| 0 | 3 | | | | | |
| | | 9 | | | | |
| 1 | 12 | | 60 | | | |
| | | 69 | | −10 | | |
| 2 | 81 | | 50 | | −254 | |
| | | 119 | | −264 | | 755 |
| 3 | 200 | | −214 | | 496 | |
| | | −100 | | 227 | | |
| 4 | 100 | | 8 | | | |
| | | −92 | | | | |
| 5 | 8 | | | | | |

$\therefore \Delta^5 f(0) = 755$

Ex) If

| $x$ : | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ : | 2 | 5 | 10 | 20 | 30 |

find by forward difference table $\Delta^4 y(1)$

Ans: $\Delta^4 f(1) = -8$

* Backward difference operator:

The operator $\nabla$ defined by $\nabla f(x) = f(x) - f(x-h)$ is called the backward difference operator.

The second backward difference of $f(x)$ denoted by $\nabla^2 f(x)$ is the backward difference of the first backward difference $\nabla f(x)$

i.e $\nabla^2 f(x) = \nabla \{\nabla f(x)\}$

$= \nabla \{ f(x) - f(x-h) \}$

$= \nabla f(x) - \nabla f(x-h)$

$= \{ f(x) - f(x-h) \} - \{ f(x-h) - f(x-2h) \}$

$\Rightarrow \nabla^2 f(x) = f(x) - 2 f(x-h) + f(x-2h)$

and so on.

In general the nth backward difference of $f(x)$ is given by $\nabla^n f(x) = \nabla^{n-1} [\nabla f(x)]$

$= \nabla^{n-1} \{ f(x) - f(x-h) \}$

$\Rightarrow \nabla^n f(x) = \nabla^{n-1} f(x) - \nabla^{n-1} f(x-h)$

* Central difference operator $\delta$:

The operator $\delta$ defined by

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

is called the central difference operator.