Ex) Show that the bisection method always converges.

Proof: Let $[p_m, q_m]$ be the interval at the mth step of bisection, having a root of the eq^n $f(x)=0$. Let x_m be the mth approximation for the root. Then initially $p_1 = a$ and $q_1 = b$

$\Rightarrow$ x_1 = first approximation $= \frac{p_1+q_1}{2}$

$$\Rightarrow p_1 < x_1 < q_1$$

Now either the root lies in $[a, x_1]$ or in $[x_1, b]$.

Therefore either $[p_2, q_2] = [p_1, x_1]$ or $[p_2, q_2] = [x_1, q_1]$.

$\Rightarrow$ either $p_2 = p_1$, $q_2 = x_1$ or $p_2 = x_1$, $q_2 = q_1$

$$\Rightarrow p_1 \leq p_2, \quad q_2 \leq q_1$$

Also, $x_2 = \frac{p_2+q_2}{2}$ so that $p_2 < x_2 < q_2$. Continuing in this way, we obtain that at the mth step

$x_m = \frac{p_m+q_m}{2}$, $p_m < x_m < q_m$ and $p_1 \leq p_2 \leq \cdots \leq p_m$

and $q_1 \geq q_2 \geq \cdots \geq q_m$.

Thus $\langle p_1, p_2, \ldots p_m, \ldots \rangle$ is bounded non-decreasing sequence (bounded by b) and $\langle q_1, q_2, \ldots, q_m, \ldots \rangle$ is a bounded non-increasing sequence (bounded by a).

Hence both these sequences converge.

Let $\lim_{m\to\infty} p_m = p$ and $\lim_{m\to\infty} q_m = q$. Now since length of the interval is decreasing at every step, we get that

$$\lim_{m\to\infty} (q_m - p_m) = 0$$

$$\Rightarrow q = p$$

Also, $p_m < x_m < q_m \Rightarrow \lim_{m\to\infty} p_m \leq \lim_{m\to\infty} x_m \leq \lim_{m\to\infty} q_m$

$$\Rightarrow p \leq \lim_{m\to\infty} x_m \leq q$$

$$\Rightarrow \lim_{m\to\infty} x_m = p = q$$

Hence $\langle x_m \rangle$ converges.

Ex) Derive $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for determining the square root of $a > 0$, using Newton-Raphson formula.

Solⁿ: let $x = \sqrt{a}$

$\Rightarrow x^2 - a = 0$

also, let $f(x) = x^2 - a = 0$

$\therefore f'(x) = 2x$

Now, from Newton-Raphson formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - a}{2x_n}$$

$$= x_n - \frac{1}{2}x_n + \frac{a}{2x_n}$$

$$= \frac{1}{2}x_n + \frac{a}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$ //

Ex) Mention two situations of Newton-Raphson formula where it fails to give a solution.

<u>Ans:-</u>

(a) Newton's method fails if $\frac{f(x_0)}{f'(x_0)}$ is not small enough. This happens, when the graph of the function $y = f(x)$ near the root is flat. Sometimes, the curve is not rather flat but a wrong choice of initial approximation leads to the failure of Newton's method.

(b) Another case of failure is the situation where two roots of the equation are close together. In this case, the simplest initial approximation x_0 is also nearly parallel to x-axis.

Ex) Perform four iterations of the Newton-Raphson method to obtain the approximate value of $(17)^{1/3}$ starting with the initial approximation $x_0 = 2$.