order of convergence:

If $x_n = a + \varepsilon_n$, $x_{n+1} = a + \varepsilon_{n+1}$, where a is a root of $f(x) = 0$ and if we able to prove that $\varepsilon_{n+1} = k\varepsilon_n^p$ where k is a constant then the order of convergence of the method is p.

Ex) Show that the rate of convergence of the Newton-Raphson formula is quadratic.

Let for the eqn $f(x)=0$, a be the exact value of the root i.e. $f(a)=0$ and let x_n be the value for nth approximation. Now let $a = x_n - \xi_n$, ξ_n is a small quantity.

$\therefore \quad x_n = a + \xi_n$

and $\quad x_{n+1} = a + \xi_{n+1}$

Now by Newton-Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow a + \xi_{n+1} = a + \xi_n - \frac{f(a+\xi_n)}{f'(a+\xi_n)}$$

$$\Rightarrow \xi_{n+1} = \xi_n - \frac{f(a+\xi_n)}{f'(a+\xi_n)} \quad \text{---(1)}$$

By Taylor's thm, we have

$$f(a+\xi_n) = f(a) + \xi_n f'(a) + \frac{\xi_n^2}{\lfloor 2} f''(a) + \cdots$$

$$= \xi_n f'(a) + \frac{\xi_n^2}{\lfloor 2} f''(a) + \cdots \quad \because f(a)=0$$

$$\text{---(2)}$$

Differentiating (2), we get,

$$f'(a+\xi_n) = f'(a) + \xi_n f''(a) + \frac{\xi_n^2}{\lfloor 2} f'''(a) + \cdots$$

$$\text{---(3)}$$

$\therefore$ (1) $\Rightarrow$

$$\xi_{n+1} = \xi_n - \frac{\xi_n f'(a) + \frac{\xi_n^2}{\lfloor 2} f''(a) + \cdots}{f'(a) + \xi_n f''(a) + \cdots}$$

$$= \xi_n - \xi_n \left[f'(a) + \frac{\xi_n}{\lfloor 2} f''(a) + \cdots \right] \left[f'(a) + \xi_n f''(a) + \cdots \right]^{-1}$$

$$= \xi_n - \xi_n \left[1 + \frac{\xi_n}{\lfloor 2} \frac{f''(a)}{f'(a)} + \cdots \right] \left[1 + \xi_n \frac{f''(a)}{f'(a)} + \cdots \right]^{-1}$$

$$= \xi_n - \xi_n \left[1 + \frac{\xi_n}{\lfloor 2} \frac{f''(a)}{f'(a)} + \cdots \right] \left[1 - \xi_n \frac{f''(a)}{f'(a)} + \cdots \right]$$

$$= \xi_n - \xi_n \left[1 - \xi_n \frac{f''(a)}{f'(a)} + \frac{\xi_n}{\lfloor 2} \frac{f''(a)}{f'(a)} - \cdots \right]$$

$$= \frac{1}{2} \xi_n^2 \frac{f''(a)}{f'(a)} \quad \left[\text{as } \xi_n \text{ is small therefore retaining terms upto } \xi_n^2 \right]$$

i.e. $\xi_{n+1} \propto \xi_n^2 \quad \left[\because \frac{f''(a)}{2f'(a)} \text{ is constant} \right]$

Hence the convergence is quadratic //

* Rate of convergence of Regula - Falsi Method:

let $x_n = a + \varepsilon_n$, $x_{n+1} = a + \varepsilon_{n+1}$ where a is a root of $f(x)=0$ and ε_n is the error involved at the nth step, when finding out the actual root a.

Now by Regula - Falsi method, we have

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n)$$

Provided that at each step $f(x_{n-1}) f(x_n) < 0$

$$\Rightarrow a + \varepsilon_{n+1} = a + \varepsilon_n - \frac{(a+\varepsilon_n) - (a+\varepsilon_{n-1})}{f(a+\varepsilon_n) - f(a+\varepsilon_{n-1})} f(a+\varepsilon_n)$$

$$\Rightarrow \varepsilon_{n+1} = \varepsilon_n - \frac{(\varepsilon_n - \varepsilon_{n-1})}{f(a+\varepsilon_n) - f(a+\varepsilon_{n-1})} f(a+\varepsilon_n)$$

$$= \frac{\varepsilon_n \{f(a+\varepsilon_n) - f(a+\varepsilon_{n-1})\} - (\varepsilon_n - \varepsilon_{n-1}) f(a+\varepsilon_n)}{f(a+\varepsilon_n) - f(a+\varepsilon_{n-1})}$$

$$\Rightarrow \varepsilon_{n+1} = \frac{\varepsilon_{n-1} f(a+\varepsilon_n) - \varepsilon_n f(a+\varepsilon_{n-1})}{f(a+\varepsilon_n) - f(a+\varepsilon_{n-1})}$$

$$= \frac{\varepsilon_{n-1}\left[f(a) + \varepsilon_n f'(a) + \frac{\varepsilon_n^2}{\lfloor 2} f''(a) + \cdots\right] - \varepsilon_n\left[f(a) + \varepsilon_{n-1} f'(a) + \frac{\varepsilon_{n-1}^2}{\lfloor 2} f''(a) + \cdots\right]}{[f(a) + \varepsilon_n f'(a)] - [f(a) + \varepsilon_{n-1} f'(a)]}$$

$$= \frac{\varepsilon_{n-1}\varepsilon_n f'(a) - \varepsilon_n \varepsilon_{n-1} f'(a) + \frac{1}{2} f''(a) \varepsilon_{n-1}\varepsilon_n (\varepsilon_n - \varepsilon_{n-1})}{\varepsilon_n f'(a) - \varepsilon_{n-1} f'(a)}$$

$$\Rightarrow \varepsilon_{n+1} = \frac{1}{2} \frac{f''(a)}{f'(a)} \varepsilon_{n-1}\varepsilon_n + O(\varepsilon_n^2) \quad \text{------ (1)}$$

[using $f(a)=0$

We shall now find a number k such that

$$\varepsilon_{n+1} = c\,\varepsilon_n^{\,k} \quad \text{and therefore}$$

$$\varepsilon_n = c\,\varepsilon_{n-1}^{\,k}$$

$$\Rightarrow \varepsilon_{n-1} = c^{-1/k}\,\varepsilon_n^{\,1/k}$$

$\because$ (1) $\Rightarrow$

$$c\,\varepsilon_n^{\,k} = l \cdot c^{-1/k}\,\varepsilon_n^{\,1/k} \cdot \varepsilon_n \quad \text{where, } l = \tfrac{1}{2} f''(a)/f'(a)$$

Hence, $k = 1 + \frac{1}{k}$, on equating powers of ε_n

$$\Rightarrow k^2 - k - 1 = 0$$

$$\Rightarrow k = \frac{1 \pm \sqrt{5}}{2}$$

The positive value of k is $\frac{1}{2}(1+\sqrt{5})$ or 1.618

Hence, $\varepsilon_{n+1} = c\,\varepsilon_n^{\,1.618}$

Thus the order of convergence of Regula-falsi method is 1.618.