

* The method of fixed point iteration or successive approximation:

To find the roots of the equation $f(x) = 0$ by the method of fixed point iteration, we rewrite it in the form

$$x = \phi(x) \quad \text{--- (1)}$$

Then if α be a root of the equation $f(x) = 0$, we must have $\alpha = \phi(\alpha)$, i.e. α is a fixed point under the mapping ϕ .

Now suppose that $[a_0, b_0]$ be the initial interval containing the root α obtained by graphical or tabulation method and the function $\phi(x)$ is continuously differentiable for sufficient number of times in $[a_0, b_0]$ such that $x \in [a_0, b_0]$, $\phi(x) \in [a_0, b_0]$ and $\phi'(x) \neq 0$ in the interval.

Let $x = x_0$ ($a_0 \leq x_0 \leq b_0$) be the initial approximation to the root α . Set $x = x_0$ on the right hand side of (1) to get the first approximation as $x_1 = \phi(x_0)$. The successive approximations are then

$$x_2 = \phi(x_1), \quad x_3 = \phi(x_2), \quad \dots, \quad x_{n+1} = \phi(x_n).$$

i.e. the iteration is generated by the formula

$$x_{n+1} = \phi(x_n) \quad \text{--- (2)}$$

x_n being the n -th approximation of the root α of $f(x) = 0$. But the sequence $\{x_n\}$ may or may not converge. If $\{x_n\}$ converges, then it must converge to α so that in the limit $\alpha = \phi(\alpha)$, i.e. α is a root of the equation (1) i.e. of $f(x) = 0$. However, the convergence of the sequence $\{x_n\}$ depends on a suitable choice of the function $\phi(x)$ and the initial approximation x_0 .

Note: Convergence of fixed point iteration:

A sufficient condition for the convergence of the sequence $\{x_n\}$ of iteration is given by the following theorem:

Let $x = \alpha$ be a root of $f(x) = 0$ and let I be an interval containing the point $x = \alpha$. Let $f(x)$ and $f'(x)$ be continuous in I where $f(x)$ is defined by the equation $x = f(x)$ which is equivalent to $F(x) = 0$. Then if $|f'(x)| < 1$ for all x in I , the sequence of approximations $x_0, x_1, x_2, \dots, x_n$ defined by $x_{n+1} = f(x_n)$ converges to the root α , provided that the initial approximation x_0 is chosen in I .

Ex) Find a real root of the equation $f(x) = x^3 + x^2 - 1 = 0$ by using iteration method.

Solⁿ: Here $f(0) = -1$ and $f(1) = 1$, so a root lies between 0 and 1. To find this root, we put the equation in the form $x = \phi(x)$.

$$\therefore x^3 + x^2 - 1 = 0 \Rightarrow x^2(1+x) = 1$$

$$\Rightarrow x^2 = \frac{1}{1+x}$$

$$\Rightarrow x = \frac{1}{\sqrt{1+x}}$$

So that $\phi(x) = \frac{1}{\sqrt{1+x}}$ and $\phi'(x) = \frac{1}{-2(1+x)^{3/2}}$

Again, $|\phi'(x)| = \left| \frac{1}{2(1+x)^{3/2}} \right| < 1$, for $0 < x < 1$

Hence the iterative method can be applied.

(*) Starting with $x_0 = 0.5$, we get

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{1+0.5}} = \frac{1}{\sqrt{1.5}} = .81649$$

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{1+.81649}} = .74196$$

$$x_3 = \phi(x_2) = \frac{1}{\sqrt{1+.74196}} = .75767$$

$$x_4 = \phi(x_3) = \frac{1}{\sqrt{1+.75767}} = .75427$$

$$x_5 = \phi(x_4) = \frac{1}{\sqrt{1+.75427}} = .75500$$

$$x_6 = \phi(x_5) = \frac{1}{\sqrt{1+.75500}} = .75485$$

$$x_7 = \phi(x_6) = \frac{1}{\sqrt{1+.75485}} = .75488$$

etc.

Ex) Compute the real root of the equation $x + \log x - 2 = 0$ correct upto three significant figures by the method of fixed point iteration.

Solⁿ: let $f(x) = x + \log x - 2$

Since $f(1) = -1 < 0$, $f(2) = 0.3010 > 0$, the positive root lies between 1 and 2.

Now the given eqⁿ can be written as

$$x = 2 - \log x = \phi(x), \text{ say}$$

Hence $\phi'(x) = -\frac{1}{x} \Rightarrow$ that $|\phi'(x)| < 1$ in $1 < x < 2$.

Then the iterative method is applicable.

Let $x_0 = 1.5$ be the initial approximation of the root of the equation. Then the successive approximations are given by

$$x_1 = \phi(x_0) = 2 - \log(1.5) = 1.5945$$

$$x_2 = \phi(1.5945) = 1.5334$$

$$x_3 = 1.5725$$

$$x_4 = 1.5473$$

$$x_5 = 1.5635$$

$$x_6 = 1.5531$$

$$x_7 = 1.5597$$

$$x_8 = 1.5555$$

$$x_9 = 1.5582$$

$$x_{10} = 1.5565$$

Thus the required root of the given eqⁿ is 1.56, correct upto three significant figures.