We can see by inspection that

$$y_1(x) = \cos x \quad \text{and} \quad y_2(x) = \sin x$$

are two solutions of the equation

$$y'' + y = 0$$

then any linear combination of these solutions, such as $y(x) = 3y_1(x) - 2y_2(x) = 3\cos x - 2\sin x$

is also a solution. We will see later that, conversely, every solution of $y'' + y = 0$ is a linear combination of these two particular solutions y_1 and y_2.

Thus a general solution of $y''+y=0$ is given by

$$y(x) = c_1 \cos x + c_2 \sin x.$$

Theorem: Existence and uniqueness for linear equations:

Suppose that the functions p, q and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)y' + q(x)y = f(x)$$

has a unique solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0 \quad , \quad y'(a) = b_1$$

We saw in the above example that $y(x) = 3\cos x - 2\sin x$ is a solution of $y''+y=0$. It has the initial values $y(0) = 3$, $y'(0) = -2$. This theorem tells us that this is the only solution with initial values. More generally, the solution $y(x) = b_0 \cos x + b_1 \sin x$ satisfies the arbitrary initial conditions $y(0) = b_0$, $y'(0) = b_1$; this illustrates the existance of such a solution, also as guaranteed by the above theorem.

Ex) Verify that the functions

$y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions of the differential equation

$$y'' - 2y' + y = 0$$ and then find a solution satisfying the initial conditions $y(0) = 3$, $y'(0) = 1$.

Soln: We have $y_1(x) = e^x$ and $y_2(x) = xe^x$

$\Rightarrow y_1'(x) = e^x$ $\quad\quad$ $\Rightarrow y_2'(x) = xe^x + e^x$

$\Rightarrow y_1''(x) = e^x$ $\quad\quad$ $\Rightarrow y_2''(x) = xe^x + e^x + e^x$

$$= xe^x + 2e^x$$

Now,

$$y_1'' - 2y_1' + y_1$$

$$= e^x - 2e^x + e^x$$

$$= 0$$

and $$y_2'' - 2y_2' + y_2$$

$$= xe^x + 2e^x - 2xe^x - 2e^x + xe^x$$

$$= 0$$

$\therefore$ $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions of the given differential equation.

Now, we impose the given initial conditions on the general solution $y(x) = c_1 e^x + c_2 x e^x$

$$\therefore \quad y'(x) = c_1 e^x + c_2 e^x + c_2 x e^x$$

$$= (c_1 + c_2)e^x + c_2 x e^x$$

Given initial conditions are

$y(0) = 3$ and $y'(0) = 1$

$\Rightarrow c_1 = 3$ $\quad\quad \Rightarrow (c_1 + c_2) = 1$

$$\therefore \quad c_2 = 1 - c_1 = 1 - 3 = -2$$

Hence the sol^n of the original initial value Problem is

$$y(x) = 3e^x - 2xe^x$$