# System of Linear Equation

Suppose, we have. the system of $m$ linear equations with $n$ variables $x_1, x_2, \ldots, x_n$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{12}x_1 + a_{12}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_n \end{aligned}$$

These equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow AX = B,$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

# Homogeneous and Non-homogeneous Linear Equations

A system of equation of the form $AX = B$, $B \neq 0$ is called a system of non-homogeneous linear equation.

If $B = 0$, the matrix equation $AX = B$ reduced to $AX = 0$, such a system of equation is called a system of homogeneous linear equations. A system of equations having one or more solution is called a consistent system of equations.

A system of equations having no solution is called an inconsistent system of equations.

## Criterion of Consistency

Let $AX = B$ be a system of $m$ linear equations in $n$ variables.

**(a)** If $|A| \neq 0$, then the system is consistent and has unique solution is given by

$$X = A^{-1}B$$

**(b)** If $|A| = 0$ and $\text{adj}(A) \cdot B = 0$, then system is consistent and has infinite number of solutions.

**(c)** If $|A| = 0$ and $(\text{adj}(A)) \cdot B \neq 0$, then system is inconsistent and has no solution.

## Working Rule to Find Solution of Linear Equations (Matrix Method)

**(i)** Write the given system of equations in matrix form $AX = B$.

**(ii)** Find $|A|$.

**(iii)** If $|A| \neq 0$, then compute $A^{-1}$ by using $A^{-1} = \frac{1}{|A|}(\text{adj}(A))$ and use the relation $X = A^{-1}B$ to find $X$.

**(iv)** If $|A| = 0$, then find $(\text{adj}(A)) \cdot B$.

**(v)** If $(\text{adj}(A)) \cdot B = 0$, then system has infinitely many solutions and if $(\text{adj}(A)) \cdot B \neq 0$, then the system has no solution.

**(vi)** If the system has infinitely many solutions, then choose a real number $k$ for one of the variable, this will reduce the number of variables by one. Now, take any two equations and solve by using matrix method.

## Solution of a System of Linear Equations by Reducing in Echelon Form

Suppose, we have a system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_n \end{aligned}$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow AX = B,$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$,

is called the coefficient matrix.

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is called the column matrix of the unknowns.

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$ is called the column matrix of the constant.

$$\text{and } C = [A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ \vdots & & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_n \end{bmatrix}$$

is called augmented matrix.

## Criteria for a System of Non-homogeneous Linear Equation $AX = B$

**(i)** If Rank $C \neq$ Rank of $A$, then the system is inconsistent has no solutions.

**(ii)** If Rank of $C$ = Rank of $A$ = number of unknowns, then the system has unique solution.

**(iii)** If Rank of $C$ = Rank of $A <$ number of unknowns, then the system has infinite number of solutions.

## For a System of Homogeneous Linear Equation $AX = 0$

A system of homogeneous equations can either the trivial solution or an infinite number of solutions.

**(i)** If rank $A = n$, number of unknowns, then the system has only the trivial solution or unique solution (each unknown equal to zero).

**(ii)** If rank $A < n$, number of unknowns, then the system has an infinite number of non-trivial solution.