

Numerical Integration

Ques) When Numerical integration is used? Explain its methods?

Sol:- It is the process of finding or evaluating a definite integral $I = \int_a^b f(x) dx$ from a set of numerical values of the integrand $f(x)$. If it is applied to the integration of a function of a single variable the process is known as quadrature. The problem of numerical integration is solved by first approximating the integrand by a polynomial with the help of an interpolation formula and then integrating this expression between the desired limits. Thus to evaluate the definite integral $\int_a^b f(x) dx$

first express the $f^n f(x)$ by an interpolation formula say $P(x)$ and then integrate $f(x)$ between the limits $[a, b]$, i.e. -

$$\int_a^b f(x) dx \sim \int_a^b P(x) dx$$

The error E in such type of approximation is given by

$$\int_a^b f(x) dx - \int_a^b P(x) dx = \int_a^b [f(x) - P(x)] dx$$

Ex) Derive the general quadrature formula of numerical integration (Newton-Cote's quadrature formula).

Proof: Let $I = \int_a^b y dx$, where $y = f(x)$ is bounded and continuous and a and b are finite. Suppose $f(x)$ is given for certain equidistant values of x say $x_0, x_0 + h, x_0 + 2h, \dots$. Let the interval $[a, b]$ be divided into n equal parts, each of width h .

do that $b-a = mh$.

Let $x_0 = a$, $x_1 = x_0 + h = a+h$, $x_2 = a+2h, \dots$
 $x_m = a+mh = b$.

We have assumed that the $m+1$ ordinates y_0, y_1, \dots

$, y_m$ are at equal intervals for every x in $[a, b]$.

At each $x_0 + nh$ let y and $xy(x)$ = f function.

$$\therefore I = \int_a^b y dx = \int_{x_0}^{x_m} y_n dx$$

or second column with following steps for each x
 $\therefore = \int_{x_0}^{x_m} y_{x_0+uh} h du$, where
 $u = \frac{x-x_0}{h}$ for making $dx = \frac{h}{1} du$ but step

of f function with h increments $\Rightarrow dn = h$ due to

$$\Rightarrow I = h \int_{0}^m \left[y_0 + u \Delta y_0 + \frac{u(u-1)}{12} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{13} \Delta^3 y_0 \right] du$$

(if) neglecting terms $- \frac{1}{4} \Delta^4 y_0$ and $\frac{1}{5} \Delta^5 y_0$ etc.

$$= h \left[my_0 + \frac{m^2}{2} \Delta y_0 + \left(\frac{m^3}{3} - \frac{m^2}{2} \right) \frac{\Delta^2 y_0}{12} + \left(\frac{m^4}{4} - m^3 + m^2 \right) \frac{\Delta^3 y_0}{13} \right. \\ \left. + \dots \text{upto } (m+1) \text{ terms} \right] \text{ actual (1) i.e.}$$

This is the general quadrature formula. We can

deduce a number of formulae from

this by putting $m=1, 2, \dots$

$$xy[(x)_1 - (x)_0] = xy(x)_1 - xy(x)_0$$

Trapezoidal Rule:

Putting $m=1$ in the formula (1) and neglecting second and higher order differences, we get

$$x_0 + h \text{ in } (x)_1 - (x)_0 = I \text{ (1) i.e.}$$

$$\int_{x_0}^{x_1} y dx = h \left[y_0 + \frac{1}{2} \Delta y_0 \right]$$

$$= h \left[y_0 + \frac{y_1 - y_0}{2} \right] = h \left[\frac{y_0 + y_1}{2} \right]$$

Similarly, $\int_{x_0+h}^{x_0+2h} y dx = h \left[\frac{y_1 + y_2}{2} \right]$

$$\dots - - - - - (x^2 + xP + x^3) \frac{d}{dx} = xbE \quad ?$$

x_0+nh

$$\int_{x_0+(n-1)h}^{x_0+nh} y dx = h \left[\frac{y_{n-1} + y_n}{2} \right]$$

$$\dots - - - - - (x^2 + xP + x^3) \frac{d}{dx} = xbE \quad ?$$

Adding these m integrals, we get,

$$\int_{x_0}^{x_0+nh} y dx = h \left[\frac{1}{2} (y_0 + y_n) + (y_1 + y_2 + \dots + y_{n-1}) \right]$$

= distance between two consecutive ordinates

$\times \frac{1}{2}$ mean of the first and the last ordinates

+ sum of all the intermediate ordinates

This rule is known as the Trapezoidal Rule.

Note: Here we have assumed that y is a function of x of the first degree, i.e., the eqn of the curve is of the form $y = ax + b$.

We can improve the accuracy of the value of the given interval by increasing the number of subintervals thereby making h very small.

* Simpson's One-third rule:

Putting $m=2$ in the formula (1) and neglecting third and higher order differences, we get,

$$\int_{x_0}^{x_0+2h} y dx = h \left[2y_0 + 2\Delta y_0 + \frac{1}{3} \left(\frac{8}{3} + 2 \right) \Delta^2 y_0 \right]$$

$$= h \left[2y_0 + 2(y_1 - y_0) + \frac{1}{3} (y_2 - 2y_1 + y_0) \right]$$

$$= \frac{h}{3} [y_0 + 4y_1 + y_2]$$

Similarly,
not 4h

$$\int_{x_0+2h} y dx = \frac{h}{3} (y_2 + 4y_3 + y_4)$$

$$\int_{x_0+mh} [y dx] = x^2 C$$

$$\int_{x_0+(m-2)h} y dx = \frac{h}{3} (y_{m-2} + 4y_{m-1} + y_m)$$

Adding $\int_{x_0} y dx + \int_{x_0+h} y dx + \int_{x_0+2h} y dx + \dots + \int_{x_0+(m-2)h} y dx = x^2 C$

$$\int_{x_0} y dx = \frac{h}{3} [(y_0 + y_m) + 4(y_1 + y_3 + \dots + y_{m-1}) + 2(y_2 + y_4 + \dots + y_{m-2})]$$

This formula is known as Simpson's one-third rule.

Note:- Here we have neglected all differences above y'' .

The second, y'' , must be a polynomial of second degree or only, so that it is "smooth".

$$y = ax^2 + bx + c$$

While using Simpson's $\frac{1}{3}$ rule, the given interval must be divided into an even number of sub-intervals.

* Simpson's three-eighth rule:

Putting $m=3$ in the formula: (1) and neglecting all differences above the third, we get

$$\int_{x_0+3h} y dx = h \left[3y_0 + \frac{9}{2} (y_1 - y_0) + \frac{(27 - 9)}{8} \frac{4^3 y_0}{12} + \frac{(81 - 27 + 9)}{8} \frac{4^3 y_0}{12} \right] d = x^3 C$$

$$= h \left[3y_0 + \frac{9}{2} (y_1 - y_0) + \frac{9}{4} (y_2 - 2y_1 + y_0) + \frac{3}{8} (y_3 - 3y_2 + 3y_1 - y_0) \right]$$

$$= \frac{3h}{8} [y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly, $x_0 + ch$

$$\int_{x_0+ch}^{x_0+3h} y dx = \frac{3h}{8} [y_3 + 3y_4 + 3y_5 + y_6]$$

$x_0 + nh$

$$\int_{x_0+(m-3)h}^{x_0+nh} y dx = \frac{3h}{8} [y_{m-3} + 3y_{m-2} + 3y_{m-1} + y_m]$$

Adding all these integrals, we have,

$x_0 + nh$

$$\int_{x_0}^{x_0+nh} y dx = \frac{3h}{8} [(y_0 + y_m) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{m-1}) + 2(y_3 + y_6 + \dots + y_{m-3})]$$

This formula is known as Simpson's three-eighth rule.

Note: Here we have neglected all differences above the third so. y is a polynomial of the third degree.

$$\text{i.e., } y = ax^3 + bx^2 + cx + d$$

while using Simpson's $\frac{3}{8}$ rule, the given interval of integration must be divided into sub-intervals whose number m is a multiple of 3.

Weddle's Rule:

Putting $m=6$ in the formula (1), we have

$$\int_{x_0}^{x_0+nh} y dx = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + (2y_7 + 5y_8 + y_9)]$$

This formula is known as [Weddle's] rule.

Note:- Here we have assumed that the fn y is of the form $y = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$.

While using Weddle's rule, the number of sub-intervals should be taken as multiple of 6.

Ex) Calculate an approximate value of $\int_0^{\pi/2} \sin x dx$ by
 (a) the Trapezoidal rule, (b) Simpson's $\frac{1}{3}$ rule,
 using 11 ordinates.

Sol: First we divide the interval $[0, \frac{\pi}{2}]$ into ten parts by taking the interval of differencing $h = \frac{\pi}{20}$
 and then we compute the values of $f(x) = \sin x$
 for each point of subdivision. These computed
 values are as shown in the following table:

$x_0 = 0$	$x_0 + h = \frac{\pi}{20} = 0.15708$	$x_0 + 2h = \frac{2\pi}{20} = 0.31416$	$x_0 + 3h = \frac{3\pi}{20} = 0.47124$	$x_0 + 4h = \frac{4\pi}{20} = 0.62832$	$x_0 + 5h = \frac{5\pi}{20} = 0.78540$	$x_0 + 6h = \frac{6\pi}{20} = 0.94248$	$x_0 + 7h = \frac{7\pi}{20} = 1.09956$	$x_0 + 8h = \frac{8\pi}{20} = 1.25664$	$x_0 + 9h = \frac{9\pi}{20} = 1.41372$	$x_0 + 10h = \frac{10\pi}{20} = 1.57080$
0	0.15708	0.31416	0.47124	0.62832	0.78540	0.94248	1.09956	1.25664	1.41372	1.57080
1	0.15708	0.31416	0.47124	0.62832	0.78540	0.94248	1.09956	1.25664	1.41372	1.57080
2	0.15708	0.31416	0.47124	0.62832	0.78540	0.94248	1.09956	1.25664	1.41372	1.57080
3	0.15708	0.31416	0.47124	0.62832	0.78540	0.94248	1.09956	1.25664	1.41372	1.57080
4	0.15708	0.31416	0.47124	0.62832	0.78540	0.94248	1.09956	1.25664	1.41372	1.57080
5	0.15708	0.31416	0.47124	0.62832	0.78540	0.94248	1.09956	1.25664	1.41372	1.57080
6	0.15708	0.31416	0.47124	0.62832	0.78540	0.94248	1.09956	1.25664	1.41372	1.57080
7	0.15708	0.31416	0.47124	0.62832	0.78540	0.94248	1.09956	1.25664	1.41372	1.57080
8	0.15708	0.31416	0.47124	0.62832	0.78540	0.94248	1.09956	1.25664	1.41372	1.57080
9	0.15708	0.31416	0.47124	0.62832	0.78540	0.94248	1.09956	1.25664	1.41372	1.57080
10	0.15708	0.31416	0.47124	0.62832	0.78540	0.94248	1.09956	1.25664	1.41372	1.57080

(a) By Trapezoidal rule, we have,

$$\begin{aligned}\int_0^{\pi/2} \sin x dx &= \frac{h}{2} \left[(y_0 + y_{10}) + 2(y_1 + y_2 + \dots + y_9) \right] \\ &= \frac{\pi}{40} \left[1.00000 + 2(5.285312) \right] \\ &= \frac{\pi}{40} [12.570624]\end{aligned}$$

which is ≈ 1.9981 which is near to $\pi/2 = 1.5708$.

(b) By Simpson's $\frac{1}{3}$ rule, we have, $I = \frac{h}{3} [y_0 + 4(y_1 + y_3 + \dots + y_9) + 2(y_2 + y_4 + \dots + y_8)]$

$$\begin{aligned}\int_0^{\pi/2} \sin x dx &= \frac{h}{3} \left[y_0 + y_{10} + 4(y_1 + y_3 + \dots + y_9) + 2(y_2 + y_4 + \dots + y_8) \right] \\ &= \frac{\pi}{60} [1.00000 + 4 \times 3.19823 + 2 \times 2.65689] \\ &= \frac{\pi}{60} \times 19.09870 = 1.0006\end{aligned}$$

Note:- The exact value of the interval

$$\int_0^{\pi/2} \sin x dx = [-\cos x]_0^{\pi/2} = 1.0000$$

∴ the error

$$\text{due to Trapezoidal rule} = 1.0000 - 0.9984$$

$$= 0.0016$$

$$\text{and due to Simpson's } \frac{1}{3} \text{ rule} = 1.0000 - 1.00066 \\ = -0.00066$$

Ex) Calculate by Simpson's $\frac{1}{3}$ rule an approximate value of $\int_{-3}^3 x^4 dx$ by taking seven equidistant ordinates. Compare it with the exact value and the value obtained by using the Trapezoidal rule.

Sol:- Divide the interval $[-3, 3]$ into six equal parts. Each sub-interval has width $h = \frac{3 - (-3)}{6} = 1$. The values of the f^n for each point of sub-division are given below:

x	$y = x^4$
$x_0 = -3$	$(-3)^4 = 81$
$x_0 + h = -2$	$(-2)^4 = 16$
$x_0 + 2h = -1$	$(-1)^4 = 1$
$x_0 + 3h = 0$	$0^4 = 0$
$x_0 + 4h = 1$	$1^4 = 1$
$x_0 + 5h = 2$	$2^4 = 16$
$x_0 + 6h = 3$	$3^4 = 81$

By Simpson's ' $\frac{1}{3}$ ' rule, we get,

$$\int_{-3}^3 x^4 dx = \frac{h}{3} \left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right] \frac{1}{2} = \\ = \frac{1}{3} [294] = 98$$

The exact value of $\int_{-3}^3 x^4 dx$ is

$$\int_{-3}^3 x^4 dx = \left[\frac{x^5}{5} \right]_{-3}^3 = \frac{1}{5} \times 486 = 97.2$$

By Trapezoidal rule, we get

$$\int_{-3}^3 x^4 dx = \frac{h}{2} [y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)]$$
$$= \frac{1}{2} \times [162 + 2 \times 34] = 115$$

In this case we observe that the Trapezoidal rule gives an inaccurate result. In general Simpson's rule gives a better result than the Trapezoidal rule.

Ex) A solid of revolution is formed by rotating about the x -axis, the area betw the x -axis, the lines $x=0$ and $x=1$ and a curve through the points with the following co-ordinates:

x	0.00	0.25	0.50	0.75	1.00	∞
y	1.0000	0.9896	0.9589	0.9089	0.8415	∞

Estimate the volume of the solid (formed by using Simpson's rule).

Sol:- Here, $h = 0.25$, $y_0 = 1.0000$, $y_1 = 0.9896$,
 $y_2 = 0.9589$, $y_3 = 0.9089$, $y_4 = 0.8415$

Then the required volume of the solid generated

$$= \int_0^1 \pi y^2 dx$$
$$= \pi \cdot \frac{h}{3} \left[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2(y_2^2) \right]$$

$$= \pi \cdot \frac{0.25}{3} [10.7687]$$

$$= 2.8192$$

Ex) Apply Simpson's $\frac{1}{3}$ rule to compute the value of π

from the formula

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$$

Sol: Divide the interval $[0, 1]$ into 6 equal parts each of width

$$\text{width } \frac{1-0}{6} = \frac{1}{6}, \text{ Hence } h = \frac{1}{6}. \text{ The values of } f(x)$$

at each point of sub-division are given below:

x	$y = \frac{1}{1+x^2}$
$x_0 = 0$	1.00000
$x_0 + h = \frac{1}{6}$	0.97297
$x_0 + 2h = \frac{2}{6}$	0.90000
$x_0 + 3h = \frac{3}{6}$	0.80000
$x_0 + 4h = \frac{4}{6}$	0.69231
$x_0 + 5h = \frac{5}{6}$	0.59016
$x_0 + 6h = 1$	0.50000

By Simpson's $\frac{1}{3}$ rule, we get,

$$\begin{aligned} \int_0^1 \frac{dx}{1+x^2} &= \frac{h}{3} \left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right] \\ &= \frac{1}{18} \left[14 \cdot [1371.63] \right] = 0.785395 \end{aligned}$$

Again,

from the question,

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}$$

$$\therefore \text{From (1) } \left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right] \frac{1}{18} = 1$$

$$\frac{\pi}{4} = 0.785395$$

$$\Rightarrow \pi = 3.14158$$

Ex) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by Simpson's one-third rule. Hence obtain the approximate value of π .

$$\text{Sol: } \int_0^1 \frac{dx}{1+x^2} = [tan^{-1}x]_0^1 = tan^{-1}1 = \frac{\pi}{4}$$

$\text{we } \frac{1}{3}$

Ex) Apply Simpson's rule to estimate the value of

the integral $\int_1^2 \frac{dx}{x}$ by dividing the interval $[1, 2]$ into four equal parts, so that $h = \frac{1}{4}$

Ans: 0.693254

Ex) A curve is drawn to pass through the points given by the following table:

$x:$	1	1.5	2	2.5	3	3.5	4
$y:$	2	2.4	2.7	2.8	3	2.6	2.1

Find the area bounded by the curve, the x -axis and the lines $x=1, x=4$.

Sol: In order to find the required area we shall compute the value of the integral

$$I = \int_1^4 y dx$$

Here, $m=6, h=0.5$

By Simpson's $\frac{1}{3}$ rule, we get,

$$\begin{aligned} I &= \frac{h}{3} \left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right] = \frac{0.5}{3} \times \\ &= \frac{0.5}{3} \left[4.1 + 4 \times 7.8 + 2 \times 5.7 \right] = \frac{1}{6} = \\ &= 7.78 \text{ units of area} \end{aligned}$$

By Weddle's Rule, we get, $I = \frac{\pi}{4}$

$$\begin{aligned} I &= \frac{3h}{10} \left[y_0 + y_6 + 5(y_1 + y_5) + 2(y_2 + y_4 + 6y_3) \right] = 7.74 \\ &= 7.74 \text{ units of area.} \end{aligned}$$

$$2P28F = \frac{\pi}{4}$$

$$321.41 \times 2 = \pi <$$

Ex) A river is 80 feet wide. The depth d (in feet) of the river at a distance x from one bank is given by the following table:

x :	0	10	20	30	40	50	60	70	80
d :	0	4	7	9	12	15	14	8	3

Find approximately the area of the cross-section of the river using Simpson's $\frac{1}{3}$ rule.

Soln: The required area of the cross-section of the river
~~starts with~~ $= \int_0^{80} d dx$

$$= \frac{h}{3} [y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{10}{3} [0 + 3 + 4(4 + 9 + 15 + 8) + 2(7 + 12 + 14)]$$

$$= \frac{10}{3} \times 213 = 710$$

Hence the required area of cross-section of the river $= 710$ sq. feet.

Ex) A rocket is launched from the ground, its accn is registered during the first 80 seconds and is given in the table below:

t (sec)	0	10	20	30	40	50	60	70	80
accn (m/sec 2)	30.00	31.63	33.44	35.47	37.75	40.33	43.25	46.69	50.57

Find the velocity of the rocket at $t = 80$ sec.

Soln: Let v denote the velocity and a denote the accn.

$$\text{Then } a = \frac{dv}{dt} [(t+10+20+30+40)] \frac{1}{5} =$$

i.e. the velocity v at $t = 80$ is given by

$$v = \int_0^{80} a dt$$

$$\text{Here } h = 10$$

By Simpson's $\frac{1}{3}$ rule, we have

$$\begin{aligned} \int_0^{80} adt &= \frac{b}{3} \left[30 \cdot 00 + 50 \cdot 67 + 4(31 \cdot 63 + 35 \cdot 47 \right. \\ &\quad \left. + 40 \cdot 33 + 46 \cdot 69) + 2(33 \cdot 44 + 37 \cdot 75 + 43 \cdot 25) \right] \\ &= \frac{10}{3} \times [80 \cdot 67 + 616 \cdot 48 + 228 \cdot 88] \\ &= 3086 \cdot 77 \end{aligned}$$

Hence velocity at time $t = 80$ is $3086 \cdot 77 \text{ m/sec}$ or 3086.77 m/sec .

* Ex) For period $0 \leq t \leq 20$ estimate the velocity of a vehicle which starts from rest, is given at fixed intervals of time t (min) as follows:

$$t : [2^0 + 4^0 + 6^0] s + 8(pt)_{0,0} + 12 + 14 + 16 + 18 + 20] \frac{d}{dt} =$$

$$v : 10 [18 + 25 + 29 + 32 + 20 + 11 + 5 + 2 + 0] \frac{d}{dt} =$$

Estimate approximately the distance covered in t (min)

~~20 minutes.~~

Sol: If s (K.m) be the distance covered in time

t (min), then we have to find s in respect of A (x)

using $\frac{ds}{dt} = v$ where v is velocity in $\frac{\text{m}}{\text{sec}}$

$$\frac{ds}{dt} = v \quad \therefore [s]_0^{20} = \int_0^{20} v dt$$

$$\begin{aligned} \therefore \cancel{\frac{ds}{dt} = v dt} \quad &\therefore [s]_0^{20} = \int_0^{20} v dt \\ &= \frac{b}{3} [(v_0 + v_{10}) + 4(v_1 + v_3 + v_5 + v_7 + v_9) + 2(v_2 + v_4 + v_6 + v_8)] \end{aligned}$$

$$= \frac{b}{3} [(0+0) + 4(10 + 25 + 32 + 11 + 2) + 2(18 + 29 + 20 + 5)]$$

$$= \frac{2}{3} [4(4) + 4(10 + 25 + 32 + 11 + 2) + 2(18 + 29 + 20 + 5)]$$

$$= \frac{2}{3} [4(4) + 4(10 + 25 + 32 + 11 + 2) + 2(18 + 29 + 20 + 5)]$$

\therefore The distance covered approximately in 20 minutes is $309 \cdot 33 \text{ K.m.}$

Ex) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i) Simpson's $\frac{3}{8}$ rule and (ii) Weddle's rule and compare the results with its actual value to find which formula gives a better approximation of the actual result.

$$[\tan^{-1} 6 = 1.4056]$$

Sol: we divide the interval $[0, 6]$ into six equal parts each of width $h=1$ and compute the values of $y = \frac{1}{1+x^2}$ at each point of subdivision. The computed values are as follows:

$$x: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$$

$$y_n: 1.00 \quad 0.500 \quad 0.200 \quad 0.100 \quad 0.058824 \quad 0.038462 \quad 0.027027$$

(i) Simpson's $\frac{3}{8}$ rule:

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8} \left[\left(y_0 + y_6 \right) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3 \right]$$

$$= \frac{3 \times 1}{8} \left[1.00 + 0.027027 \right] = 1.3571$$

(ii) Weddle's rule:

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10} \left[2y_0 + 5y_1 + 2y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 \right]$$

$$= 1.3734$$

Also, $\int_0^6 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^6 = \tan^{-1} 6 = 1.4056$

Hence the value of the integral found by Weddle's rule is the nearest to the actual value followed by its value given by Simpson's $\frac{3}{8}$ rule.

Ex) Calculate the value of the integral

$$\int_{4}^{5.2} \log x \, dx \text{ by } (a) \text{ Trapezoidal rule, } (b) \text{ Simpson's } \frac{1}{3} \text{ rule}$$

(c) Simpson's $\frac{3}{8}$ rule and (d) Weddle's rule.

(a) Trapezoidal rule, (b) Simpson's $\frac{1}{3}$ rule

(c) Simpson's $\frac{3}{8}$ rule (d) Weddle's rule.

After finding the true value of the integral, compare

the errors in the four cases.

Sol: Taking $h=0.2$ divide the interval $[4, 5.2]$ into six equal parts. The values of $\log x$ for each point of sub-division are given below:

$$x_0 = 4.0 \quad y_0 = 1.3862944 \quad ; \quad \frac{1}{8} \text{ deg. } (i)$$

$$x_1 = 4.2 \quad y_1 = 1.4350845 \quad ; \quad \frac{1}{8} \text{ deg. } (i)$$

$$x_2 = 4.4 \quad y_2 = 1.491716045 \quad ; \quad \frac{1}{8} \text{ deg. } (i)$$

$$x_3 = 4.6 \quad y_3 = 1.5269563 \quad ; \quad \frac{1}{8} \text{ deg. } (ii)$$

$$x_4 = 4.8 \quad y_4 = 1.5686159 \quad ; \quad \frac{1}{8} \text{ deg. } (ii)$$

$$x_5 = 5.0 \quad y_5 = 1.6094379 \quad ; \quad \frac{1}{8} \text{ deg. } (ii)$$

$$x_6 = 5.2 \quad y_6 = 1.6486586 \quad ; \quad \frac{1}{8} \text{ deg. } (ii)$$

(a) By Trapezoidal rule, we have

$$\int_{4}^{5.2} \log x \, dx = \frac{h}{2} [y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= 1.8276551$$

(b) By Simpson's $\frac{1}{3}$ rule, we have

$$\int_{4}^{5.2} \log x \, dx = \frac{h}{3} [y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= 1.8278472$$

(c) By Simpson's $\frac{3}{8}$ rule, we have

$$\int_0^{5.2} e^x dx = \frac{3h}{8} [y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= 1.827847$$

(d) By Weddle's rule, we have

$$\int_0^{5.2} e^x dx = \frac{3h}{10} [y_0 + y_6 + 5(y_1 + y_5) + 2(y_2 + y_4 + 6y_3)]$$

$$= \frac{3 \times 0.2}{10} \times 30.464123$$

$$= 1.8278474$$

Actual value of $\int_0^{5.2} e^x dx = [e^{(x)}]_0^{5.2} = 1.8278475$.

$$\int_0^{5.2} e^x dx = \frac{(e^{5.2} - e^0)}{5.2} = 1.8278475$$

Hence the errors due to different formulae are

(a) -0.0001924 (b) $+0.0000003$

(c) -0.0000005 (d) $+0.0000001$

We observe that Weddle's rule is more accurate.

Ex) Evaluate $\int_0^4 e^x dx$ by {Simpson's Rule} given that

$e^x = 2.72$, $e^4 = 54.60$ and compare it with the actual value.

Sol: Divide $[0, 4]$ into 4 equal parts by taking $h=1$.

By Simpson's $\frac{1}{3}$ rule, $\int_0^4 e^x dx = \frac{h}{3} [y_0 + 4(y_1 + y_3) + 2y_2]$

$$\int_0^4 e^x dx = 53.87$$

Actual value of $\int_0^4 e^x dx = e^4 - e^0 = 53.60$

$$\int_0^4 e^x dx = \frac{h}{3} [y_0 + 4(y_1 + y_3) + 2y_2]$$

$$\int_0^4 e^x dx = \frac{1}{3} \cdot 2(e^1 - e^0)$$

$$(e^1 - e^0) \approx 1 = 1$$