

$$(x-x_0)(x-x_1)(x-x_2) + (x-x_1)(x-x_2)(x-x_3) + \dots$$

\* Central Differences and Gauss Interpolation Formulae:

The (central) difference formulae are used for interpolating values of the function near the middle of a tabulated set.

The operator  $\delta$ , called the central difference operator, is defined by the operator  $\delta^m$

$$\delta^m = f(x+h) - f(x-h)$$

$$= f(x+mh) - f(x-mh)$$

The first central difference  $\delta f(x)$  is given by

$$\delta f(x) = f(x+\frac{1}{2}h) - f(x-\frac{1}{2}h)$$

$$= \frac{1}{h}(f(x+h) - f(x)) - \frac{1}{h}(f(x-h) - f(x))$$

$$= \frac{1}{h}[f(x+h) - 2f(x) + f(x-h)]$$

$$= \frac{1}{h}[f(x+h) - f(x)] - \frac{1}{h}[f(x) - f(x-h)]$$

$$= \left( E^{\frac{1}{2}} - E^{-\frac{1}{2}} \right) f(x)$$

$$\therefore \delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

$$= E^{-\frac{1}{2}} (E - 1)$$

$$= E^{-\frac{1}{2}} \Delta \quad [\because E \equiv 1 + \Delta]$$

Again, we have  $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

$$= E^{\frac{1}{2}} (1 - E^{-1})$$

$$= E^{\frac{1}{2}} \Delta \quad [\because E^{-1} \equiv 1 - \Delta]$$

$$\therefore \delta = E^{-\frac{1}{2}} \Delta = E^{\frac{1}{2}} \Delta$$

The higher power of the operator  $\delta$  applied to  $y_n$  gives

$$\text{Assume } \delta^m y_n = \Delta^m E^{-\frac{m}{2}} y_n = \nabla^m E^{\frac{m}{2}} y_n \text{ for } m \geq 1$$

$$\Rightarrow \delta^m y_n = \Delta^m y_{n-\frac{m}{2}} = \nabla^m y_{n+\frac{m}{2}+1}$$

\* Gauss's central difference formula:

(a) Gauss's forward formula:

The general Newton's formula is

$$y = f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$

$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + (x-x_0)(x-x_1)(x-x_2)$$

$$(x-x_3) f(x_0, x_1, x_2, x_3, x_4) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

$$f(x_0, x_1, x_2, x_3, x_4, x_5) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)$$

$$+ f(x_0, x_1, x_2, x_3, x_4, x_5, x_6) + \dots \text{ up to } (1) \text{ term}$$

Putting  $x_0 = x_0$ ,  $x_1 = x_0 + h$ ,  $x_2 = x_0 - h$ ,  $x_3 = x_0 + 2h$ ,  $x_4 = x_0 - 2h$ ,  
 $x_5 = x_0 + 3h$ ,  $x_6 = x_0 - 3h$  etc. we get,

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_0+h) + (x-x_0)(x-x_0-h) f(x_0, x_0+h, x_0-h)$$

$$+ (x-x_0)(x-x_0-h)(x-x_0+h) f(x_0, x_0+h, x_0-h, x_0+2h)$$

$$+ (x-x_0)(x-x_0-h)(x-x_0+h)(x-x_0-2h) f(x_0, x_0+h, x_0-h, x_0+2h, x_0-2h)$$

$$+ (x-x_0)(x-x_0-h)(x-x_0+h)(x-x_0+2h)(x-x_0+2h) \times$$

$$f(x_0, x_0+h, x_0-h, x_0+2h, x_0-2h, x_0+3h) + \dots$$

$$\text{Now, put } u = \frac{x - x_0}{h}$$

$$\Rightarrow x - x_0 = uh$$

$$f(x) = f(x_0) + hu f(x_0, x_0 + h) + hu(hu - h)f(x_0 - h, x_0, x_0 + h) \\ + hu(hu - h)(hu + h)f(x_0 - h, x_0, x_0 + h, x_0 + 2h) \\ + hu(hu - h)(hu + h)(hu - 2h)f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) \\ + hu(hu - h)(hu + h)(hu - 2h)f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) \\ + hu(hu - h)(hu + h)(hu - 2h)f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) \\ + \dots$$

But, we have,  $f(x_0, x_0 + h) = \frac{\Delta y_0}{h}$ ,  $f(x_0 - h, x_0, x_0 + h) = \frac{\Delta y_{-1}}{h}$

$$f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^3 y_{-1}}{3! h^3}$$

$$f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^4 y_{-2}}{4! h^4}$$

$$f(x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h) = \frac{\Delta^5 y_{-2}}{5! h^5}$$

Substituting these values in (2), we get,  $(d + u)d$  and  $(d + u)d$

$$y = y_0 + hu \frac{\Delta y_0}{h} + h^2 u(u-1) \frac{\Delta^3 y_{-1}}{2! h^3} + h^3 u(u-1)(u+1) \frac{\Delta^3 y_{-1}}{3! h^3}$$

$$+ h^4 u(u-1)(u+1)(u-2) \frac{\Delta^4 y_{-2}}{4! h^4} + h^5 u(u-1)(u+1)(u-2)(u+2)$$

$$\frac{\Delta^5 y_{-2}}{5! h^5} + \dots$$

$$y = y_0 + u \Delta y_0 + u(u-1) \frac{\Delta^3 y_{-1}}{2! (d + u)d} + u(u-1) \frac{\Delta^3 y_{-1}}{3! (d + u)d} \\ + u(u-1)(u-2) \frac{\Delta^4 y_{-2}}{4! (d + u)d} + u(u-1)(u-2)(u-4) \frac{\Delta^5 y_{-2}}{5! (d + u)d} + \dots$$

This result is known as Gauss's forward formula.

for equal intervals.

The result (3) may also be written as

$$y = y_0 + u c_1 \Delta y_0 + u c_2 \Delta^3 y_{-1} + u^2 c_3 \Delta^3 y_{-1} + u^3 c_4 \Delta^4 y_{-2} \\ + u^4 c_5 \Delta^5 y_{-2} + \dots$$

(b) Gauss's backward formula:

Substituting

$$x_0 = x_0, x_1 = x_0 - h, x_2 = x_0 - 2h, x_3 = x_0 - 3h, x_4 = x_0 - 4h,$$

$x_5 = x_0 - 5h, x_6 = x_0 - 6h$ , etc. in the general Newton-Rule formula (1), we get

$$\begin{aligned} y &= y_0 + u(x-x_0) f(x_0, x_0-h, x_0+0) \\ &\quad + (x-x_0)(x-x_0+h) f(x_0, x_0-h, x_0+0, x_0+2h) \\ &\quad + (x-x_0)(x-x_0+h)(x-x_0-h) f(x_0, x_0-h, x_0+0, x_0+2h) \\ &\quad + (x-x_0)(x-x_0+h)(x-x_0-h)(x-x_0+2h) f(x_0, x_0-h, x_0+0, \\ &\quad x_0+2h, x_0+4h) + (x-x_0)(x-x_0+h)(x-x_0-h)(x-x_0+2h) \\ &\quad (x-x_0-2h) f(x_0, x_0-h, x_0+0, x_0+2h, x_0+4h, x_0+6h) + \dots \end{aligned}$$

$$\text{Now, put } u = \frac{x-x_0}{h} \Rightarrow x-x_0 = uh \quad (\text{we get, } u \in \mathbb{R} \text{ and } u \in \mathbb{R})$$

$$\Rightarrow x-x_0 = uh \quad (\text{we get, } u \in \mathbb{R} \text{ and } u \in \mathbb{R})$$

$$\begin{aligned} y &= y_0 + hu f(x_0-h, x_0) + hu(hu) f(x_0-h, x_0, x_0+0) \\ &\quad + hu(hu) (hu-h) f(x_0-2h, x_0-h, x_0, x_0+0) \\ &\quad + hu(hu) (hu-h) (hu+2h) f(x_0-2h, x_0-h, x_0, x_0+0, x_0+2h) \\ &\quad + hu(hu) (hu-h) (hu+2h) (hu-2h) \times \dots + \dots \\ &= (1+u) f(x_0-3h, x_0-2h, x_0-h, x_0, x_0+0, x_0+2h) + \dots \end{aligned} \quad (4)$$

But, we know that

$$f(x_0-h, x_0) = \frac{\Delta y_{-1}}{h}, \quad f(x_0-h, x_0, x_0+0) = \frac{\Delta^2 y_{-1}}{2! h^2},$$

$$f(x_0-2h, x_0-h, x_0, x_0+0) = \frac{\Delta^3 y_{-2}}{3! h^3},$$

$$f(x_0-2h, x_0-h, x_0, x_0+0, x_0+2h) = \frac{\Delta^4 y_{-2}}{4! h^4}$$

$$f(x_0-3h, x_0-2h, x_0-h, x_0, x_0+0, x_0+2h) = \frac{\Delta^5 y_{-3}}{5! h^5} \quad \text{etc.}$$

Substituting these in (4), we get, the required formula

$$\begin{aligned} y &= y_0 + u \Delta y_{-1} + u(u+1) \frac{\Delta^2 y_{-1}}{2!} + u(u+1)(u+2) \frac{\Delta^3 y_{-2}}{3!} \\ &\quad + u(u+1)(u+2)(u+3) \frac{\Delta^4 y_{-2}}{4!} + u(u+1)(u+2)(u+3)(u+4) \frac{\Delta^5 y_{-3}}{5!} + \dots \end{aligned} \quad (5)$$

This result is known as Gauss's backward formula.

The formula (5) may also be written as

$$y = y_0 + u c_1 \Delta y_1 + u+1 c_2 \Delta^2 y_{-1} + u+1 c_3 \Delta^3 y_{-2} + u+2 c_4 \Delta^4 y_{-2}$$
$$+ u+2 c_5 \Delta^5 y_{-3} + \dots$$

(e) Third formula due to Gauss:

$$y = y_1 + (u-1) \Delta y_0 + u(u-1) \frac{\Delta^2 y_0}{2!} + u(u-1) \frac{\Delta^3 y_{-1}}{3!}$$
$$+ u(u-1)(u-2) \frac{\Delta^4 y_{-1}}{4!} + u(u-1)(u-2)(u-3) \frac{\Delta^5 y_{-2}}{5!} + \dots$$

Note: The central difference formulae are more appropriate when the value of the argument for unknown entry lies somewhere in the middle of the difference table. The basic reason is that the central difference formulae are likely to converge more rapidly.

The Gauss forward or backward formulae are suitable according as the interpolated value lies near to the right or left of the central value in the table.

Bessel's interpolation formula is used for arguments  $x$  near the middle of the table and the number of given arguments is even. no record in column odd.

Again Stirling's formula is used when the interpolating point  $x$  is near the centre of the table and the number of arguments is odd.

$$\text{Stirling's formula: } y = y_0 + \frac{\Delta y_0}{2!} (x-x_0) + \frac{\Delta^2 y_0}{4!} (x-x_0)^2 + \dots$$
$$+ \frac{\Delta^3 y_0}{6!} (x-x_0)^3 + \frac{\Delta^4 y_0}{8!} (x-x_0)^4 + \dots$$

\* Stirling's Formula: Gauss's forward formula and Gauss's backward formula

we have,

Gauss's forward formula

$$y = y_0 + u \Delta y_0 + u(u-1) \frac{\Delta^2 y_0}{12} + u(u-1)(u-2) \frac{\Delta^3 y_1}{13} + u(u-1)(u-2)(u-3) \frac{\Delta^4 y_2}{14} + u(u-1)(u-2)(u-3)(u-4) \frac{\Delta^5 y_3}{15} + \dots$$

and Gauss's backward formula

$$y = y_0 + u \Delta y_{-1} + u(u+1) \frac{\Delta^2 y_{-1}}{12} + u(u-1) \frac{\Delta^3 y_{-2}}{13} + u(u-1)(u+2) \frac{\Delta^4 y_{-3}}{14} + u(u-1)(u-2)(u-3) \frac{\Delta^5 y_{-4}}{15} + \dots$$

Taking the mean of the two Gauss's formulae, we have

$$\begin{aligned} y_u &= y_0 + u \left( \frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{u}{12} \Delta^2 y_{-1} \\ &\quad + \frac{u(u-1)}{13} \cdot \frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u(u-1)(u-2)}{14} \Delta^4 y_{-2} \\ &\quad + \frac{u(u-1)(u-2)(u-3)}{15} \cdot \frac{(\Delta^5 y_{-3} + \Delta^5 y_{-4})}{2} \end{aligned}$$

where  $u = \frac{n-n_0}{h}$

This formula is known as Stirling's formula.

\* Bessel's Formula:

we have,

Gauss's forward formula

$$\begin{aligned} y_r &= y_0 + u \Delta y_0 + u(u-1) \frac{\Delta^2 y_0}{12} + u(u-1)(u-2) \frac{\Delta^3 y_1}{13} \\ &\quad + u(u-1)(u-2) \frac{\Delta^4 y_2}{14} + u(u-1)(u-2)(u-3) \frac{\Delta^5 y_3}{15} + \dots \end{aligned}$$

and third formula due to Gauss

$$\begin{aligned} y &= y_1 + (u-1) \Delta y_0 + u(u-1) \frac{\Delta^2 y_0}{12} + u(u-1)(u-2) \frac{\Delta^3 y_2}{13} \\ &\quad + u(u-1)(u-2) \frac{\Delta^4 y_1}{14} + u(u-1)(u-2)(u-3) \frac{\Delta^5 y_2}{15} + \dots \end{aligned}$$

Taking the mean of Gauss's forward formula and the

third formula due to Gauss, we obtain,

$$y_u = \frac{y_0 + y_1}{2} + (u - \frac{1}{2}) \Delta y_0 + \frac{u(u-1)}{2} \frac{(\Delta^2 y_{-1} + \Delta^2 y_0)}{2} +$$

$$+ \frac{u(u-\frac{1}{2})(u-1)}{3} \frac{\Delta^3 y_{-1}}{2} + \frac{u(u-1)(u-2)}{4} \frac{(\Delta^4 y_{-2} + \Delta^4 y_{-1})}{2} +$$

$$+ \frac{u(u-\frac{1}{2})(u-1)(u-2)}{5} \Delta^5 y_{-2} + \dots$$

where  $(u-1) u = \frac{n-n_0}{h}$

This formula is known as Bessel's formula.

Ex) Find by Gauss's backward formula the sales by a

concern for the year 1936 given below

Year AB PEPY 1901 1911 1921 1931 1941 1951

Sale (in thousands): 12 15 20 27 39 52

S.I.:- Taking 1931 as the origin and  $n=10$ , then  
sales of the concern is to be found for year 1936

$$u = \frac{n-n_0}{h} = \frac{1936-1931}{10} = 0.5$$

The difference table is given below

x	$u = \frac{n-1931}{10}$	$y_u$	$\Delta y_u$	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
1901	-3	12	12	12	12	12	12
1911	-2	15	3	2	1	0	0
1921	-1	20	5	2	0	0	0
1931	0	27	7	3	1	0	0
1941	1	39	12	5	3	1	-1
1951	2	52	13	1	-4	5	10

Again, Gauss's backward formula is

$$y_u = y_0 + u c_1 \Delta y_{-1} + u c_2 \Delta^2 y_{-2} + u c_3 \Delta^3 y_{-3} + u c_4 \Delta^4 y_{-4} + u c_5 \Delta^5 y_{-5}$$

$$= y_0 + u \Delta y_{-1} + \frac{u(u+1)}{2} \Delta^2 y_{-2} + \frac{(u+1)u(u-1)}{6} \Delta^3 y_{-3}$$

$$+ \frac{(u+2)(u+1)u(u-1)}{24} \Delta^4 y_{-4} + \frac{(u+2)(u+1)u(u-1)(u-2)}{120} \Delta^5 y_{-5}$$

$$\therefore y_{0.5} = 27 + 0.5 \times 7 + \frac{(0.5)(1.5)}{2} \times 5 + \frac{(1.5)(0.5)(-0.5)}{6} \times 3$$

$$+ \frac{(2.5)(1.5)(-0.5)}{24} \times (-7) + \frac{(2.5)(1.5)(0.5)(-0.5)(-1.5)}{120} \times (-10)$$

$$= 27 + 3.5 + 1.875 - 1.875 + 27.34 - 117.18$$

$$= 32.3437 \text{ (approx)}$$

**Ex)** Using Gauss's backward formula, estimate the population of Latur town for the year 1974 from the following data:

Year	1939	1949	1959	1969	1979	1989
Population	12.2	15.2	20.2	27.4	39.52	52
(in thousands)	12.2	15.2	20.2	27.4	39.52	52

$$2.0 = \frac{1974 - 1939}{10} = \frac{35 - 30}{10} = n$$

**Ex)** Use Gauss's forward formula to find  $y_{30}$  given

that

$$y_{21} = 18.4708; y_{25} = 17.8144; y_{29} = 17.1070;$$

$$y_{33} = 16.3432; y_{37} = 15.5151$$

Sol:- Take  $x_0 = 29$ ,  $h = 4$

We require the value of  $y$  for  $x = 30$

$$\text{i.e. for } u = \frac{x-x_0}{h} = \frac{30-29}{4} = 0.25$$

The difference table is given below:

$x$	$u$	$y_u$	$\Delta y_u$	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
21	-2	18.4708				
25	-1	17.8144	-0.6564	-0.051		
29	0	17.1070	-0.7074	-0.0564	-0.0054	-0.0022
33	1	16.3432	-0.7638	-0.064	-0.0076	
37	2	15.5154	-0.8278			

Now, Gauss's forward formula is

$$\begin{aligned}
 y_u &= y_0 + u c_1 \Delta y_0 + u c_2 \frac{\Delta^2 y_{-1}}{2} + u c_3 \frac{\Delta^3 y_{-1}}{6} + u c_4 \frac{\Delta^4 y_{-2}}{24} \\
 &= y_0 + u \Delta y_0 + \frac{u(u-1)}{2} \Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{6} \Delta^3 y_{-1} \\
 &\quad + \frac{u(u+1)(u-1)(u-2)}{24} \Delta^4 y_{-2}
 \end{aligned}$$

$$\begin{aligned}
 \therefore y_{0.25} &= 17.1070 + 0.25 \times (-0.7638) + (0.25) \frac{(-0.0564)}{2} \\
 &\quad + \frac{(0.25)(1+0.25)(0.25-1)}{6} (-0.0076) \\
 &\quad + \frac{(0.25)(0.25+1)(0.25-1)(0.25-2)}{24} (-0.0022) \\
 &\doteq 16.9216 + (\text{approx}) + \left( \frac{1-\Delta + \Delta}{11} \right) u + \dots = u
 \end{aligned}$$

Ex) Given that  $\sqrt{12500} = 111.803399$ ;  $\sqrt{12510} = 111.848111$ ;

$$\sqrt{12520} = 111.89280615 - \frac{\sqrt{12530}}{s} = 111.937483$$

Show by Gauss's backward formula that

$$\sqrt{12516} = 111.874930 + \frac{(1-\Delta)(1+\Delta)(1-\Delta)(1-\Delta)}{11} s$$

Ex) Use Stirling's formula to find  $y_{28}$  given

$$y_{20} = 49225, y_{25} = 48316, y_{30} = 47236,$$

$$y_{35} = 45926, y_{40} = 44306$$

Sol: Taking  $x_0 = 30$  as the origin and  $h = 5$  as the unit, we are to find the value of  $y_{28}$  for

$$u = \frac{x - x_0}{h} = \frac{28 - 30}{5} = -0.4$$

The difference table is given below:

$x$	$u = \frac{x - 30}{5}$	$y_u$	$\Delta y_u$	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$
20	-2	49225				
25	-1	48316	-171			
30	0	47236	-10805	-59		
35	1	45926	-230	-121		
40	2	44306	-1310	-80		

The  $\rightarrow$  Stirling's formula ( $\text{in } u = \frac{x - 30}{5}$ ) +

$$y_u = y_0 + u \left( \frac{\Delta y_0 + \Delta y_1}{2} \right) + \left( \frac{u^2}{2^2} \Delta^2 y_{-1} + \frac{u(u+1)(u-1)}{24} \Delta^4 y_{-2} \right) + \dots$$

$$\left( \frac{\Delta^3 y_1 + \Delta^3 y_2}{2} \right) + \frac{u^3(u+1)(u-1)}{24} \Delta^6 y_{-3} + \dots$$

$$\begin{aligned} y_{-0.4} &= 47236 + (-0.4) \left\{ \frac{-1310 - 10805}{2} \right\} + \frac{(-0.4)^2}{2} (-230) \\ &+ \frac{(-0.4)(-0.4+1)(-0.4-1)}{6} \left\{ \frac{-80 - 59}{2} \right\} + \dots \\ &+ \frac{(-0.4)^3(-0.4+1)(-0.4-1)}{24} (-21) = 47641.82 \text{ (approx)} \end{aligned}$$

Ex) Apply Bessel's formula to obtain  $y_{25}$ , given

$$y_{20} = 2854, y_{24} = 3162, y_{28} = 3544, y_{32} = 3992$$

Soln:- Take 24 as the origin and 4 as the unit.

We are to find  $y_u$  for the value of

$$u = \frac{25-24}{4} = 0.25$$

The difference table is

$n$	$u = \frac{n-24}{4}$	$y_n$	$\Delta y_n$	$\Delta^2 y_n$	$\Delta^3 y_n$
20	-1	2854			
24	0	3162	308	74	
28	1		382	-8	
32	2	3544	66		
			448		
		3992			

Now, Bessel's formula is

$$y_u = \frac{1}{2}(y_0 + y_1) + (u - \frac{1}{2})\Delta y_0 + \frac{u(u-1)}{12} \left( \frac{\Delta^2 y_1 + \Delta^2 y_0}{2} \right) + \frac{(u - \frac{1}{2})u(u-1)}{3!} \Delta^3 y_1 + \dots$$

$$\begin{aligned} y_{0.25} &= \frac{1}{2}(3162 + 3544) + \left(\frac{1}{4} - \frac{1}{2}\right)382 + \frac{\frac{1}{4}(\frac{1}{4}-1)}{2} \left( \frac{74 + 66}{2} \right) \\ &\quad + \frac{\left(\frac{1}{4} - \frac{1}{2}\right)\frac{1}{4}(\frac{1}{4}-1)}{3!} (-8) \\ &= 3250.875 \text{ (approx)} \end{aligned}$$

Ex) Given that  $y_{20} = 24, y_{24} = 32, y_{28} = 35, y_{32} = 40,$

Find  $y_{25}$  by Bessel's formula.

Ams:- 32.9452 (approx)