

Theorem:- Divided differences are symmetric fn's of their arguments i.e. the value of any divided difference is independent of the order of the arguments.

Proof: Let $f(x_0), f(x_1), \dots, f(x_n)$ be the values of the function $y = f(x)$ corresponding to the values x_0, x_1, \dots, x_n of the argument x . Then, we have

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{-[f(x_0) - f(x_1)]}{-(x_0 - x_1)} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= f(x_1, x_0)$$

Also, $f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_1)}{x_1 - x_0} - \frac{f(x_0)}{x_1 - x_0}$

$$= \frac{f(x_1) + f(x_0)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1}$$

$$= \sum \frac{f(x_0)}{x_0 - x_1}$$

which is symmetric fn of arguments x_0 and x_1 .

Thus $f(x_0, x_1) = f(x_1, x_0)$

Again, $f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$

$$= \frac{1}{x_2 - x_0} \left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right]$$

$$= \frac{f(x_2) - f(x_1)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_2 - x_0)(x_1 - x_0)}$$

$$= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

$$= \sum \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

which is symmetric fn of arguments x_0, x_1 and x_2

Thus, $f(x_0, x_1, x_2) = f(x_1, x_2, x_0) = f(x_2, x_1, x_0)$ etc.
we can use any permutation of the arguments x_0, x_1, x_2 for the notation of second order divided difference.

Proceeding like this, by mathematical induction we can prove that $f(x_0, x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n, x_0)$
 $= f(x_2, x_3, \dots, x_n, x_0, x_1) = f(\text{any one of the permutations of the arguments } x_0, x_1, \dots, x_n)$.

This proves that the divided differences are symmetric functions of their arguments.

Ex) Find the polynomial of the lowest possible degree which assumes the values 3, 12, 15, -21 when x has the value 3, 2, 1, -1 respectively.

Sol:- The divided difference table is

x	$f(x)$	$\Delta f(x)$ $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$	$\Delta^2 f(x)$ $\frac{\Delta f(x_1) - \Delta f(x_0)}{x_2 - x_1}$	$\Delta^3 f(x)$ $\frac{\Delta^2 f(x_1) - \Delta^2 f(x_0)}{x_3 - x_2}$
-1	-21	$\frac{(-1-x_0) f(x_1) - (-1-x_0) f(x_0)}{x_1 - x_0}$ $\frac{15+21}{1+1} = 18$	$\frac{-3-18}{2+1} = -7$	
1	15	$\frac{(1-x_0) f(x_2) - (1-x_0) f(x_1)}{x_2 - x_1}$ $\frac{12-15}{2-1} = -3$	$\frac{-9+3}{3+1} = -3$	$\frac{3+7}{3+1} = 1$
2	12	$\frac{(2-x_0) f(x_3) - (2-x_0) f(x_2)}{x_3 - x_2}$ $\frac{3-12}{3-2} = -9$		
3	3			

Now applying Newton's divided difference formula; we get

$$\begin{aligned}
 f(x) &= f(x_0) + f(x_0, x_1)(x-x_0) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) \\
 &\quad + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3) + \dots \\
 &= -21 + \{x+1\}18 + (x+1)(x-1)(-7) + (x+1)(x-1)(x-2)-1 \\
 &= -21 + 18x + 18 - 7x^2 + 7 + x^3 - 2x^2 - x + 2 \\
 f(x) &= x^3 - 9x^2 + 17x + 6
 \end{aligned}$$

which is the reqd. polynomial.

* Lagrange's interpolation formula:

Let $f(x_0), f(x_1), f(x_2), \dots, f(x_m)$ be the $(m+1)$ values of

the function $y = f(x)$ corresponding to the arguments $x_0, x_1, x_2, \dots, x_m$ respectively.

It is assumed that the $f(x)$ is a polynomial of in x and since $(m+1)$ values of $f(x)$ are given so $(m+1)$ th difference is zero. Thus $f(x)$ is supposed to be polynomial in x of degree m .

Let

$$f(x) = A_0 (x-x_1)(x-x_2)\dots(x-x_{m-1})(x-x_m) + A_1 (x-x_0)(x-x_2)\dots(x-x_m) + \dots + A_m (x-x_0)(x-x_1)\dots(x-x_{m-1}) \quad (1)$$

where A_i 's are constants.

To find $A_0, A_1, A_2, \dots, A_m$ we put

$$x = x_0, x_1, \dots, x_m \text{ respectively in (1).}$$

Thus putting $x = x_0$ in (1), we get,

$$f(x_0) = A_0 (x_0-x_1)(x_0-x_2)\dots(x_0-x_m) + 0 + 0 + \dots$$

$$\Rightarrow A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_m)}$$

Similarly, by putting $x = x_1$, we get,

$$A_1 = \frac{f(x_1) - f(x_0)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_m)} \text{ and so on}$$

Thus,

$$A_m = \frac{f(x_n) - f(x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{m-1})}$$

Substituting these values of $A_0, A_1, A_2, \dots, A_m$ in (1) we get

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_m)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_m)} f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_m)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_m)} f(x_1) + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{m-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{m-1})} f(x_n) \quad (2)$$

This is called Lagrange's interpolation formula and can be used for both equal and unequal intervals.

* Advantage and disadvantage of Lagrangian interpolation:

Advantages:

- The formula is simple and easy to remember.
- There is no need to construct the divided differences table and we can directly interpolate the unknown value with the help of observation.
- There is no restriction on spacing and order of the tabulating points x_0, x_1, x_2, \dots etc.

Disadvantages:

- The calculations in the formula are more complicated.
- If we want to increase the degree of the interpolating polynomial by one more tabular point, the computations are to be made afresh; the previous calculations are of little help.

Ex) Given:

$$\log_{10} 654 = 2.8156, \quad \log_{10} 658 = 2.8182, \quad \log_{10} 659 = 2.8189,$$

$$\log_{10} 661 = 2.8202$$

Find by using Lagrange's formula, the value of $\log_{10} 656$

Sol:- Here, $x_0 = 654, x_1 = 658, x_2 = 659, x_3 = 661$

$$\text{and } f(x) = \log_{10} 656 - 3x + 2.8156 + 2.8182 - \dots$$

∴ By Lagrange's formula we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$\Rightarrow \log_{10} 656 = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times 2.8156 + () \\ + () + () = 2.8170$$

$$\therefore \log_{10} 656 = 2.8170 //$$

Ex) Construct the divided difference table and hence determine $f(x)$ as a polynomial in x for the following data:

$$n: -4 \quad -1 \quad 0 \quad 2 \quad 5$$

$$f(x): 1245 \quad 33 \quad 5 \quad 9 \quad 1335$$

Ams:- $f(x) = 13x^4 - 5x^3 + 6x^2 - 14x + 5$

Ex) Applying Lagrange's formula to obtain the value of $f(9)$ using the following table:

$$n: 5 \quad 7 \quad 11 \quad 13 \quad 17$$

$$f(x): 150 \quad 392 \quad 1452 \quad 2366 \quad 5202$$

Ams:- 810

Ex) By means of Lagrange's formula, prove that

$$y_1 = y_3 - 3(y_5 - y_3) + 2(y_3 - y_5)$$

Sol:- We are required to obtain y_1 while y_5, y_3, y_5 , and y_3 are given. By Lagrange's formula we have

$$\begin{aligned} y_1 &= \frac{(1+3)(1+3)(1-5)}{(-5+3)(-5-3)(-5-5)} y_{-5} + \frac{(1+5)(1+3)(1-5)}{(-3+5)(-3-3)(-3-5)} y_{-3} \\ &\quad + \frac{(1+5)(1+3)(1-5)}{(3+5)(3+3)(3-5)} y_3 + \frac{(1+5)(1+3)(1-5)}{(5+5)(5+3)(5-3)} y_5 \\ &= -\frac{y_5}{5} + 2\frac{y_3}{2} + y_3 - \frac{3y_5}{10} \\ &= -0.2y_5 + 0.5y_3 + y_3 - 0.3y_5 \\ &= -0.2y_5 + (0.3+0.2)y_3 + y_3 - 0.3y_5 \end{aligned}$$

$$\Rightarrow y_1 = \frac{y_3 - 3(y_5 - y_3) + 2(y_3 - y_5)}{(1+3)(1+3)(1-5)} = 0.2$$

$$\begin{aligned} &= \frac{(y_3 - 3y_5)(1-5)}{(1+3)(1+3)(1-5)} + 2 \cdot \frac{(y_3 - y_5)(1-5)}{(1+3)(1+3)(1-5)} \\ &= \frac{(y_3 - 3y_5)(-4)}{(1+3)(1+3)(1-5)} + 2 \cdot \frac{(y_3 - y_5)(-4)}{(1+3)(1+3)(1-5)} \\ &= -0.2y_5 + 0.2y_3 + 0.2y_3 - 0.2y_5 \end{aligned}$$

* Relation betⁿ divided differences and ordinary differences:

Let the given fⁿ be $y = f(x)$, let $f(x_0), f(x_1), \dots, f(x_n)$

be the values of the fⁿ $f(x)$ corresponding to the arguments
 $x_0, x_1, x_2, \dots, x_n$. $\Delta f(x) = f(x_1) - f(x_0)$ etc.

Suppose that the arguments x_0, x_1, \dots, x_m are equally spaced

and let $h = x_1 - x_0 = x_2 - x_1 = \dots = x_m - x_{m-1}$

$$\therefore x_2 - x_0 = [x_3 - x_1] = \dots = 2h$$

$$x_3 - x_0 = x_4 - x_1 = \dots = 3h \text{ and so on.}$$

$$\text{Now, } f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{h} = \frac{\Delta f(x_0)}{h}$$

$$\text{and } f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = \frac{\Delta f(x_1) / h - \Delta f(x_0) / h}{2h}$$

$$= \frac{\Delta f(x_1) + \Delta f(x_0)}{2h} = \frac{\{f(x_2) - f(x_1)\} - \{f(x_1) - f(x_0)\}}{2h}$$

$$= \frac{f(x_2) - 2f(x_1) + f(x_0)}{2h} = \frac{\Delta^2 f(x_0)}{2!h^2}$$

$$\text{Similarly, } f(x_0, x_1, x_2, x_3) = \frac{\Delta^3 f(x_0)}{3!h^3} \text{ and so on.}$$

$$\text{In general, } (xf(x_0, x_1, x_2, \dots, x_m)) = x \frac{\Delta^m f(x_0)}{m!h^m} + \dots \quad //$$

$$(x(x-x_0)(x-x_1)(x-x_2) \dots (x-x_m)) f(x_0, x_1, x_2, \dots, x_m) \text{ etc. etc.}$$