

Theorem:

Consider the differential equation

$$M(x, y) dx + N(x, y) dy = 0 \quad \dots \dots (1)$$

where M and N have continuous first partial derivatives at all points (x, y) in a rectangular domain D .

(i) If the differential equation (1) is exact in D ,

$$\text{then } \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \quad \dots \dots (2) \text{ for all } (x, y) \in D.$$

(ii) Conversely if $\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$ for all $(x, y) \in D$, then the differential equation (1) is exact in D .

Proof:

Part (i): If the differential equation (1) is exact in D , then $M dx + N dy$ is an exact differential in D . By definition of an exact differential, there exists a function F such that

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y) \quad \text{for all } (x, y) \in D.$$

$$\text{Then } \frac{\partial^2 F(x, y)}{\partial y \partial x} = \frac{\partial M(x, y)}{\partial y} \quad \text{and} \quad \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial N(x, y)}{\partial x} \quad \text{for all } (x, y) \in D.$$

But, using the continuity of the first partial derivatives of M and N , we have,

$$\frac{\partial^2 F(x, y)}{\partial y \partial x} = \frac{\partial^2 F(x, y)}{\partial x \partial y}$$

$$\text{and therefore } \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \quad \text{for all } (x, y) \in D.$$

Part (ii)

This being the converse of Part (i), we shall start with the hypothesis that

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x} \quad \text{for all } (x, y) \in D, \text{ and}$$

set out to show that $M dx + N dy = 0$ is exact in D .

This means that we must prove that there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \dots (3)$$

$$\text{and } \frac{\partial F(x,y)}{\partial y} = N(x,y) \quad \dots (4)$$

for all $(x,y) \in D$.

Let us assume that F satisfies (3). Then

$$F(x,y) = \int M(x,y) dx + \phi(y) \quad \dots (5)$$

where $\int M(x,y) dx$ indicates a partial integration w.r.t x , holding y constant, and ϕ is an arbitrary function of y only. This $\phi(y)$ is needed in (5) so that $F(x,y)$ given by (5) will represent all solutions of (3).

Differentiating (5) partially w.r.t y we obtain

$$\frac{\partial F(x,y)}{\partial y} = \frac{\partial}{\partial y} \int M(x,y) dx + \frac{d\phi(y)}{dy}$$

Now, if (4) is to be satisfied, we must have

$$N(x,y) = \frac{\partial}{\partial y} \int M(x,y) dx + \frac{d\phi(y)}{dy} \quad \dots (6)$$

$$\text{and hence } \frac{d\phi(y)}{dy} = N(x,y) - \frac{\partial}{\partial y} \int M(x,y) dx$$

Since ϕ is a function of y only, the derivative $\frac{d\phi}{dy}$ must also be independent of x . That

is, in order for (6) to hold,

$$N(x,y) - \frac{\partial}{\partial y} \int M(x,y) dx \quad \dots (7)$$

must be independent of x .

we shall show that

$$\frac{\partial}{\partial x} \left[N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right] = 0$$

we have,

$$\frac{\partial}{\partial x} \left[N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right] = \frac{\partial N(x, y)}{\partial x} - \frac{\partial^2}{\partial x \partial y} \int M(x, y) dx$$

If (3) and (4) are to be satisfied, then using the hypothesis (1), we must have

$$\frac{\partial^2}{\partial x \partial y} \int M(x, y) dx = \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial^2 F(x, y)}{\partial y \partial x} = \frac{\partial^2}{\partial y \partial x} \int M(x, y) dx$$

Thus we obtain

$$\frac{\partial}{\partial x} \left[N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right] = \frac{\partial N(x, y)}{\partial x} - \frac{\partial^2}{\partial y \partial x} \int M(x, y) dx$$

and hence

$$\frac{\partial}{\partial x} \left[N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right] = \frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y}$$

But by the hypothesis (1), $\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$

for all $(x, y) \in D$.

Thus $\frac{\partial}{\partial x} \left[N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right] = 0$ for all $(x, y) \in D$

and so (7) is independent of x . Thus we

may write

$$\phi(y) = \int \left[N(x, y) - \int \frac{\partial M(x, y)}{\partial y} dx \right] dy$$

Substituting this into equation (5), we have

$$F(x, y) = \int M(x, y) dx + \int \left[N(x, y) - \int \frac{\partial M(x, y)}{\partial y} dx \right] dy \quad \dots (8)$$

This $F(x, y)$ thus satisfies both (3) and (4) for all $(x, y) \in D$ and so $M dx + N dy = 0$ is exact in D .