Ex) Solve:

$$(x+2y+3)\,dx + (2x+4y-1)\,dy = 0 \quad \text{-----}(1)$$

Soln: Here $a_1 = 1,\ b_1 = 2,\ c_1 = 3$

$a_2 = 2,\ b_2 = 4,\ c_2 = -1$

and

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = 2$$

We therefore let

$$z = x + 2y$$

$$\therefore (1) \Rightarrow (z+3)\,dx + (2z-1)\left(\frac{dz - dx}{2}\right) = 0$$

$$\Rightarrow (2z+6)\,dx + (2z-1)(dz - dx) = 0$$

$$\Rightarrow 2z\,dx + 6\,dx + 2z\,dz - 2z\,dx - dz + dx = 0$$

$$\Rightarrow 7\,dx + (2z-1)\,dz = 0$$

which is separable. Integrating, we have

$$7x + z^2 - z = c$$

Replacing z by $x+2y$, the reqd. soln is

$$7x + (x+2y)^2 - (x+2y) = c \quad //$$

Ex) Solve: $\dfrac{dy}{dx} = \dfrac{x+2y-3}{2x+y-3}$

Soln: Take

$$\left.\begin{aligned} x &= X+h \\ y &= Y+k \\ \text{so that } \frac{dy}{dx} &= \frac{dY}{dX} \end{aligned}\right\} \quad \text{---} (1)$$

$\therefore$ Given equation becomes

$$\frac{dY}{dX} = \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \quad \text{---} (2)$$

choose h, k so that

$$\left.\begin{aligned} h+2k-3 &= 0 \\ \text{and } 2h+k-3 &= 0 \end{aligned}\right\} \quad \text{---} (3)$$

Solving (3) we get, $h=1,\ k=1$

$\therefore (1) \Rightarrow \quad X = x-1, \quad Y = y-1$

$$\therefore (2) \Rightarrow \frac{dY}{dX} = \frac{X+2Y}{2X+Y} = \frac{1+2\left(\frac{Y}{X}\right)}{2+\frac{Y}{X}} \quad \text{---} (4)$$

Take $\dfrac{Y}{X} = v \Rightarrow Y = vX$

$$\Rightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$$

$\therefore$ (4) $\Rightarrow$

$$v + x\frac{dv}{dx} = \frac{1+2v}{2+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+2v}{2+v} - v = \frac{1-v^2}{2+v}$$

$$\Rightarrow \frac{dx}{x} = \frac{(2+v)\,dv}{(1-v)(1+v)}$$

$$= \left[\frac{1}{2}\left(\frac{1}{1+v}\right) + \frac{3}{2}\left(\frac{1}{1-v}\right)\right] dv$$

Integrating,

$$\log x + \log c = \frac{1}{2}\left[\log(1+v) - 3\log(1-v)\right]$$

$$\Rightarrow 2\log(cx) = \log\left\{\frac{1+v}{(1-v)^3}\right\}$$

$$\Rightarrow x^2c^2 = \frac{1+v}{(1-v)^3}$$

$$\Rightarrow x^2c^2(1-v)^3 = 1+v$$

$$\Rightarrow x^2c^2\left(1-\frac{Y}{X}\right)^3 = 1+\frac{Y}{X}$$

$$\Rightarrow c^2(X-Y)^3 = X+Y$$

$$\Rightarrow c^2\{x-1-(y-1)\}^3 = x-1+y-1$$

$$\Rightarrow c^2(x-y)^3 = x+y-2$$

$$\Rightarrow c'(x-y)^3 = x+y-2 \quad // \text{ where } c' = c^2$$