Bernoulli's equation:

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n \quad \text{-----(1)}$$

where P and Q are constants or functions of x alone and n is constant except 0 and 1, is called a Bernoulli's differential equation.

We first multiply by y^{-n}, thereby expressing it in the form $f'(y)\frac{dy}{dx} + Pf(y) = Q$.

$$\therefore \quad y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \quad \text{-----(2)}$$

Let $y^{1-n} = v$

Diff. w.r.t x, we get

$$(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx} \quad \text{-----(3)}$$

$$\therefore \quad (2) \Rightarrow \quad \frac{1}{(1-n)}\frac{dv}{dx} + Pv = Q$$

$$\Rightarrow \frac{dv}{dx} + P(1-n)v = Q(1-n)$$

which is linear in v and x. Its I.F. $= e^{\int P(1-n)dx}$

$$= e^{(1-n)\int P dx}$$

and hence the reqd. soln is

$$v \cdot e^{(1-n)\int P dx} = \int Q(1-n)e^{(1-n)\int P dx}dx + c$$

$$\Rightarrow y^{1-n}e^{(1-n)\int P dx} = \int Q(1-n)e^{(1-n)\int P dx}dx + c$$

Note: $\frac{dx}{dy} + P_1 x = Q_1 x^n$ is also in the Bernoulli's form. Here P_1 and Q_1 are functions of y alone or constants.

Ex) Solve $x\left(\frac{dy}{dx}\right) + y = y^2 \log x$

Sol^n: Given eq^n can be written as

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = \frac{1}{x}\log x \quad \cdots (1)$$

Putting $y^{-1} = v \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore$ (1) $\Rightarrow$

$$-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x}\log x$$

$$\Rightarrow \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}\log x \quad \cdots (2)$$

Here $I.F. = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log\left(\frac{1}{x}\right)} = \frac{1}{x}$

$\therefore$ Reqd. sol^n is

$$v \cdot \frac{1}{x} = -\int \frac{1}{x}\log x \cdot \frac{1}{x}\,dx + c$$

$$\Rightarrow v x^{-1} = -\int x^{-2} \cdot \log x \, dx + c$$

$$= -\left[\log x \frac{x^{-1}}{-1} - \int \frac{1}{x} \frac{x^{-1}}{-1}\,dx\right] + c$$

$$= -\left[-x^{-1}\log x + \int x^{-2}\,dx\right] + c$$

$$= +x\log x - \frac{x^{-1}}{(-1)} + c$$

$$= +x^{-1}\log x + \frac{1}{x} + c$$

$$\Rightarrow v = +\log x + 1 + cx$$

$$\Rightarrow y^{-1} = \log x + 1 + cx$$

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Ex) Solve: $(x^3y^2 + xy)\,dx = dy$

Sol^n: Given eq^n can be written as

$$\frac{dy}{dx} = x^3y^2 + xy$$

$$\Rightarrow \frac{dy}{dx} - xy = x^3y^2$$

$$\Rightarrow y^{-2}\frac{dy}{dx} - xy^{-1} = x^3 \quad \text{---}(1)$$

Putting $y^{-1} = v \Rightarrow -y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore$ (1) $\Rightarrow$

$$-\frac{dv}{dx} - xv = x^3$$

$$\Rightarrow \frac{dv}{dx} + xv = -x^3$$

Here $I.F. = e^{\int x\,dx} = e^{x^2/2}$

$\therefore$ Reqd. sol^n is

$$v.\,e^{x^2/2} = \int -x^3.\,e^{x^2/2}\,dx + c$$

$$= -2\int t\,e^t\,dt + c$$

[Put $\frac{x^2}{2} = t$ $\Rightarrow x\,dx = dt$]

$$= -2\left[t.e^t - \int e^t\,dt\right] + c$$

$$= -2te^t + 2e^t + c$$

$$= -2e^t(t-1) + c$$

$$\Rightarrow y^{-1}e^{x^2/2} = -2e^{x^2/2}\left(\frac{x^2}{2} - 1\right) + c$$

$$\Rightarrow y^{-1} = -2\left(\frac{x^2}{2} - 1\right) + ce^{-x^2/2}$$

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