Linear Differential Equations:

A first order differential equation is called linear if it can be written in the form

$$\frac{dy}{dx} + Py = Q \quad \text{-----}(1)$$

where P and Q are constants, or functions of x alone (i.e. not of y).

To solve equation (1), we multiply both sides of equation (1) by $e^{\int P\,dx}$ (called integrating factor) so that equation (1) reduces to

$$\frac{d}{dx}\left[y.e^{\int P\,dx}\right] = Q\,e^{\int P\,dx}$$

on integrating both sides, we get,

$$y.e^{\int P\,dx} = \int \left(Q.e^{\int P\,dx}\right) dx + c$$

$$\Rightarrow \quad y.(I.F) = \int \{Q.(I.F)\}\,dx + c$$

where $I.F = e^{\int P\,dx}$

Note: Two formulas $e^{m\log A} = A^m$ and $e^{-m\log A} = \frac{1}{A^m}$ will be often used in simplifying I.F.

Remarks: Sometimes a differential equation cannot be put in the form (1) of a linear equation. Then we regard y as the independent variable and x as the dependent variable and obtain a differential equation of the form

$$\frac{dx}{dy} + P_1 x = Q_1 \quad \text{---}(2)$$

where P_1 and Q_1 are constants or functions of y alone. In this case,

$$I.F = e^{\int P_1\, dy}$$

and the required solution is

$$x \cdot (I.F) = \int \{ Q_1 \cdot (I.F) \} \, dy + c$$

Ex) Solve: $x(x-1)\frac{dy}{dx} - (x-2)y = x^2(2x-1)$

Sol^n:- we have,

$$\frac{dy}{dx} - \frac{x-2}{x(x-1)} y = \frac{x^2(2x-1)}{x(x-1)}$$

$$\Rightarrow \frac{dy}{dx} - \frac{(x-2)}{x(x-1)} y = \frac{x(2x-1)}{(x-1)}$$

Here $I.F = e^{\int -\frac{(x-2)}{x(x-1)} dx}$

$$= e^{\int \left(\frac{1}{x-1} - \frac{2}{x}\right) dx}$$

$$= e^{\log(x-1) - 2\log x}$$

$$= e^{\log\left(\frac{x-1}{x^2}\right)} = \frac{x-1}{x^2}$$

Hence, reqd. solⁿ is

$$y \cdot \frac{x-1}{x^2} = \int \frac{x(2x-1)}{(x-1)} \times \frac{x-1}{x^2}\, dx + c$$

$$= \int \left(\frac{2x-1}{x}\right) dx + c$$

$$= \int \left(2 - \frac{1}{x}\right) dx + c$$

$$y \cdot \frac{x-1}{x^2} = 2x - \log x + c$$

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Ex) Solve: $x\cos x \left(\frac{dy}{dx}\right) + y\,(x\sin x + \cos x) = 1$

Solⁿ: we have

$$\frac{dy}{dx} + y\left(\frac{x\sin x + \cos x}{x\cos x}\right) = \frac{1}{x\cos x}$$

$$\Rightarrow \frac{dy}{dx} + y\left(\tan x + \frac{1}{x}\right) = \frac{\sec x}{x}$$

$$\therefore \quad I.F. = e^{\int \left(\tan x + \frac{1}{x}\right) dx}$$

$$= e^{\log \sec x + \log x}$$

$$= e^{\log(x \sec x)} = x\sec x$$

Hence the reqd. solⁿ is

$$y \cdot x\sec x = \int \left(\frac{\sec x}{x} \cdot x\sec x\right) dx + c$$

$$= \int \sec^2 x\, dx + c$$

$$\Rightarrow y \cdot x\sec x = \tan x + c$$

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