POISSON DISTRIBUTION

6.1. DERIVATION OF THE DISTRIBUTION

Poisson Distribution was discovered by a French mathematician S.D. Poisson in 1837.

This distribution is the limiting form of the Binomial distribution when :

(*i*) the number of trials n becomes sufficiently large *i.e.*, $n \to \infty$.

(*ii*) the probability p of success in a trial is very small *i.e.*, $p \to 0$.

(*iii*) $np = \lambda$ (say) is a positive constant.

Suppose X is a binomial random variable with parameters (n, p) *i.e.*, $b\,(x\,;\,n, p)$

$$P(X = x) = p(x) = {}^nC_x \, p^x \, q^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!}\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)\,(n-2)\,\ldots\,(n-\overline{x-1})\,\lambda^x}{n^x \; x!} \; \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$\therefore \quad \lim_{n\to\infty} p(x) = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)\frac{\lambda^x}{x!}\;\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n \qquad \left[\because \text{ As } n \to \infty, \frac{1}{n} \to 0\right]$$

$$= \frac{e^{-\lambda} \lambda^x}{x!} \qquad \left[\because \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}\right]$$

which is the p.m.f. of Poisson distribution.

6.1.1. Mathematical Form

A discrete random variable X is said to follow Poisson distribution if its p.m.f. is given by

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} \; ; \; x = 0, 1, 2, 3, \ldots\ldots$$

This distribution has one parameter. λ. $X \sim P(\lambda)$ means that X is a Poisson variate with parameter λ.

6.2. EXAMPLES OF POISSON DISTRIBUTION

Poisson distribution is applicable where there are number of random situations where the probability of a success on a single trial is small and the number of trials is large. Some examples of random variables that follows Poisson probability law are:

(i) The number of misprints on a page of a book.

(ii) Number of telephone calls received in a telephone exchange in some unit of times during a day.

(iii) The number of air accidents in a country in one year.

(iv) The number of cars passing through a busy street per unit time interval during a day.

(v) The number of deaths in a city in one year by a rare disease.

(vi) The number of defective material in a large packing manufactured by a reputed company etc.

6.3 MEAN AND VARIANCE OF POISSON DISTRIBUTION

$$\begin{aligned}
\text{Mean} &= E(X) \\
&= \Sigma x p(x) \\
&= \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!} \\
&= \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x(x-1)!} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}
\end{aligned}$$

$$= \lambda e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right]$$

$$= \lambda e^{-\lambda} \,.\, e^{\lambda}$$

$$= \lambda \,.\, e^{0}$$

$$= \lambda \qquad [\because e^0 = 1]$$

$$\text{Variance} = V(X)$$

$$= E(X^2) - \{E(X)\}^2$$

$$= E[X(X-1) + X] - \{E(X)\}^2$$

$$= E[X(X-1) + E(X)] - \{E(X)\}^2$$

Now,

$$E[X(X-1)] = \sum x\,(x-1)\;p(x)$$

$$= \sum_{x=0}^{\infty} x\,(x-1)\,\frac{e^{-\lambda}\,\lambda^x}{x!}$$

$$= \sum_{x=2}^{\infty} x\,(x-1)\,\frac{e^{-\lambda}\,\lambda^x}{x\,(x-1)\,(x-2)!}$$

$$= \lambda^2\,e^{-\lambda}\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!}$$

$$= \lambda^2\,e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right]$$

$$= \lambda^2\,e^{-\lambda}\,.\,e^{\lambda}$$

$$= \lambda^2 e^0$$

$$= \lambda^2$$

$$V(X) = \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

$$\text{Mean} = \text{Variance} = \lambda$$