13.1 Properties of a Vector Field: Divergence and Curl

IN THIS SECTION definition of a vector field, divergence, curl

DEFINITION OF A VECTOR FIELD

The satellite photograph in Figure 13.1 shows wind measurements over the Atlantic Ocean. Wind direction is indicated by directed line segments. This is an example of a *vector field*, in which every point in a given region of the plane or space is assigned a vector. Here is the definition of a vector field in $\mathbb{R}^3$.

Vector Field

A **vector field** in $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns a vector to each point in its domain. A vector field with domain D in $\mathbb{R}^3$ has the form

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

where the scalar functions M, N, and P are called the **components** of $\mathbf{F}$. A **continuous** vector field $\mathbf{F}$ is one whose components M, N, and P are continuous.

Figure 13.1 A wind-velocity map of the Atlantic Ocean

For example,

$$\mathbf{F} = 2x^2y\mathbf{i} + e^{yz}\mathbf{j} + \left(\tan\frac{x}{2}\right)\mathbf{k}$$

is a vector field with **i**-component $2x^2y$, **j**-component e^{yz}, and **k**-component $\tan\frac{x}{2}$.

A vector field in $\mathbb{R}^2$ can be thought of as a special case where there are no z-coordinates and no **k**-components. That is, a vector field in $\mathbb{R}^2$ has the form

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

To visualize a particular vector field $\mathbf{F}(x, y, z)$, it often helps to select a number of points in the domain of $\mathbf{F}$ and then draw an arrow emanating from each point $P(a, b, c)$ with the direction of $\mathbf{F}(a, b, c)$ and length representing the magnitude $\|\mathbf{F}(a, b, c)\|$. We will refer to such a representation as the **graph of F**. Here is an example involving the graph of a vector field in $\mathbb{R}^2$.

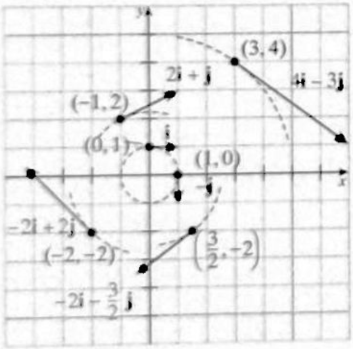


Figure 13.2 The graph of the vector field $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

EXAMPLE 1 Graph of a vector field

Sketch the graph of the vector field $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$.

Solution

We will evaluate $\mathbf{F}$ at various points. For example,

$$\mathbf{F}(3, 4) = 4\mathbf{i} - 3\mathbf{j} \quad \text{and} \quad \mathbf{F}(-1, 2) = 2\mathbf{i} - (-1)\mathbf{j} = 2\mathbf{i} + \mathbf{j}$$

We can generate as many such vector values of $\mathbf{F}$ as we wish. Several are shown in Figure 13.2. ∎

The graph of a vector field often yields useful information about the properties of the field. For instance, suppose $\mathbf{F}(x, y)$ represents the velocity of a compressible fluid (like a gas) at a point (x, y) in the plane. Then $\mathbf{F}$ assigns a velocity vector to each point in the plane, and the graph of $\mathbf{F}$ provides a picture of the fluid flow. Thus, the flow in Figure 13.4a is a constant, whereas Figure 13.4b suggests a circular flow.

Gravitational, electrical, and magnetic vector fields play an important role in physical applications. We will discuss gravitational fields now, and electrical and magnetic fields later in this section. Accordingly, we begin with Newton's law of gravitation.

TECHNOLOGY NOTE

The examples in Figure 13.3 show some vector fields obtained using computer software.

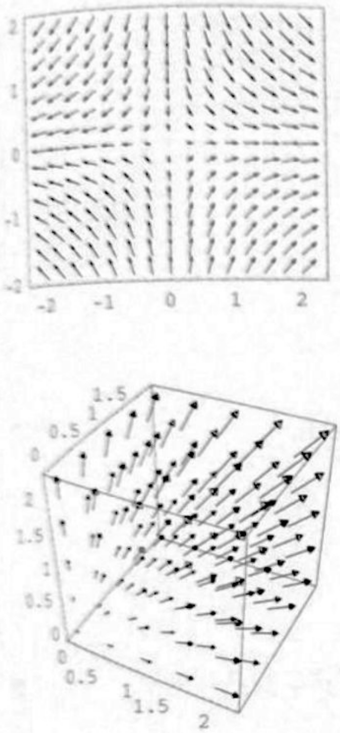
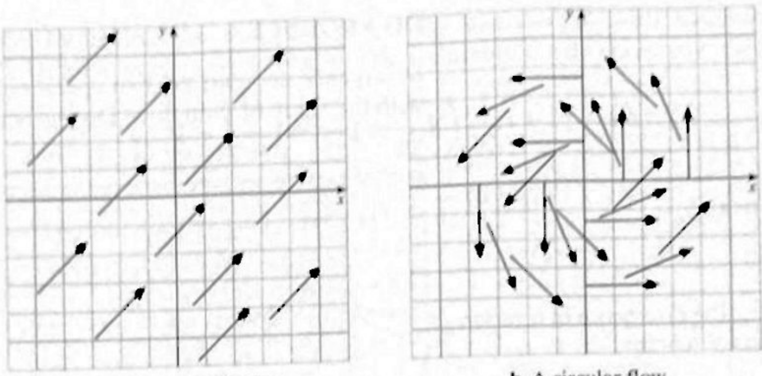


Figure 13.3 Computer generated vector fields



a. A constant fluid flow

b. A circular flow

Figure 13.4 Flow diagrams

which says that a point mass (particle) m at the origin exerts on a unit point mass located at the point $P(x, y, z)$ a force $\mathbf{F}(x, y, z)$ given by

$$\mathbf{F}(x, y, z) = \frac{Gm}{x^2 + y^2 + z^2}\mathbf{u}(x, y, z)$$

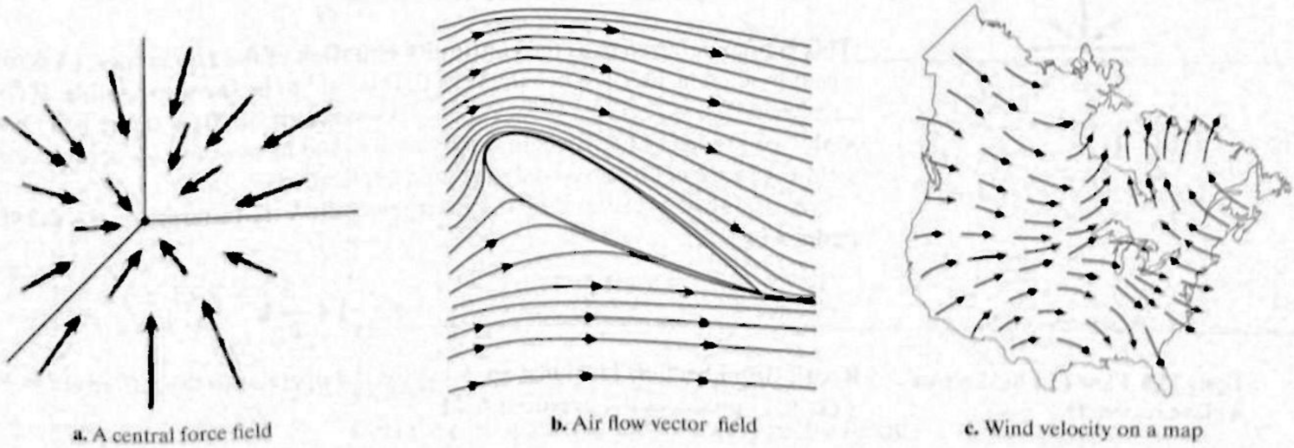
where G is a constant (the universal gravitational constant) and $\mathbf{u}$ is the unit vector extending from the point P toward the origin. The vector field $\mathbf{F}(x, y, z)$ is called the **gravitational field** of the point mass m. Because

$$\mathbf{u}(x, y, z) = \frac{-1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

it follows that

$$\mathbf{F}(x, y, z) = \frac{-Gm}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

Note that the gravitational field $\mathbf{F}$ always points toward the origin and has the same magnitude for any point m located $r = \sqrt{x^2 + y^2 + z^2}$ units from the origin. Such a vector field is called a **central force field**. This force field is shown in Figure 13.5a. Other physical vector fields are shown in Figures 13.5b and 13.5c.



a. A central force field

b. Air flow vector field

c. Wind velocity on a map

Figure 13.5 Examples of physical vector fields

DIVERGENCE

Divergence and *curl* are two operations on vector fields that originated in connection with the study of fluid flow. Divergence may be defined as follows.

Divergence

WARNING The divergence of a vector field is a scalar function.

> The **divergence** of a vector field
>
> $$\mathbf{V}(x, y, z) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$$
>
> is denoted by div $\mathbf{V}$ and is given by
>
> $$\operatorname{div} \mathbf{V} = \frac{\partial u}{\partial x}(x, y, z) + \frac{\partial v}{\partial y}(x, y, z) + \frac{\partial w}{\partial z}(x, y, z)$$

EXAMPLE 2 Divergence of a vector field

Find the divergence of each of the following vector fields.

a. $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^3\mathbf{j}$
b. $\mathbf{G}(x, y, z) = x\mathbf{i} + y^3z^2\mathbf{j} + xz^3\mathbf{k}$

Solution

a. $\operatorname{div} \mathbf{F} = \dfrac{\partial}{\partial x}(x^2y) + \dfrac{\partial}{\partial y}(xy^3) = 2xy + 3xy^2$

b. $\operatorname{div} \mathbf{G} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y^3z^2) + \dfrac{\partial}{\partial z}(xz^3) = 1 + 3y^2z^2 + 3xz^2$ ■

Suppose the vector field

$$\mathbf{V}(x, y, z) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$$

represents the velocity of a fluid with density $\delta(x, y, z)$ at a point (x, y, z) in a certain region R in $\mathbb{R}^3$. Then the vector field $\delta\mathbf{V}$ is called the **flux density** and is denoted by $\mathbf{D}$. We can think of $\mathbf{D} = \delta\mathbf{V}$ as measuring the "mass flow" of the liquid.

Assuming there are no external processes acting on the fluid that would tend to create or destroy fluid, it can be shown that div $\mathbf{D}$ gives the negative of the time rate change of the density, that is,

$$\operatorname{div} \mathbf{D} = -\frac{\partial \delta}{\partial t}$$

This is often referred to as the **continuity equation** of fluid dynamics. (A derivation is given in Section 13.7.) When div $\mathbf{D} = 0$, $\mathbf{D}$ is said to be **incompressible**. If div $\mathbf{D} > 0$ at a point (x_0, y_0, z_0), the point is called a **source**; if div $\mathbf{D} < 0$, the point is called a **sink** (see Figure 13.6). The terms *sink*, *source*, and *incompressible* apply to any vector field $\mathbf{F}$ and are not reserved only for fluid applications.

A useful way to think of the divergence div $\mathbf{V}$ is in terms of the **del operator** defined by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

Recall (from Section 11.6) that applying the del operator to the differentiable function $f(x, y, z)$ produces the **gradient field**

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

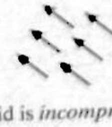
div $\mathbf{D} > 0$

Fluid flows from a *source point.*

div $\mathbf{D} < 0$

Fluid flows toward a *sink point.*

div $\mathbf{D} = 0$



Fluid is *incompressible.*

Figure 13.6 Flow of a fluid across a plane region, $\mathbf{D}$

Similarly, by taking the dot product of the operator ∇ with the vector field $\mathbf{V} = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$, we obtain the divergence

$$\begin{aligned}\nabla \cdot \mathbf{V} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \\ &= \frac{\partial}{\partial x}(u) + \frac{\partial}{\partial y}(v) + \frac{\partial}{\partial z}(w) \\ &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\ &= \operatorname{div} \mathbf{V}\end{aligned}$$

CURL

The del operator may also be used to describe another derivative operation for vector fields, called the *curl*.

Curl

> The **curl** of a vector field
>
> $$\mathbf{V}(x, y, z) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$$
>
> is denoted by curl $\mathbf{V}$ and is defined by
>
> $$\operatorname{curl} \mathbf{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}$$

Note that

$$\begin{aligned}\operatorname{curl} \mathbf{V} &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \begin{matrix} \leftarrow \text{Standard basis vectors} \\ \leftarrow \nabla \\ \leftarrow \mathbf{V} \end{matrix} \\ &= \nabla \times \mathbf{V}\end{aligned}$$

The determinant form of curl $\mathbf{V}$ is a convenient device for remembering the definition and is helpful in organizing computations.

> **Del Operator Forms for Divergence and Curl**
>
> Consider a vector field
>
> $$\mathbf{V}(x, y, z) = u(x, y, z)\mathbf{i} + v(x, y, z)\mathbf{j} + w(x, y, z)\mathbf{k}$$
>
> The divergence and curl of $\mathbf{V}$ are given by
>
> $$\operatorname{div} \mathbf{V} = \nabla \cdot \mathbf{V} \quad \text{and} \quad \operatorname{curl} \mathbf{V} = \nabla \times \mathbf{V}$$

WARNING Notice that div $\mathbf{V}$ is a scalar, and curl $\mathbf{V}$ is a vector.

EXAMPLE 3 Curl of a vector field

Find the curl of each of the following vector fields:

$$\mathbf{F} = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k} \quad \text{and} \quad \mathbf{G} = (x \cos y)\mathbf{i} + xy^2\mathbf{j}$$

Solution

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}xyz^2 - \frac{\partial}{\partial z}xy^2z\right)\mathbf{i} - \left(\frac{\partial}{\partial x}xyz^2 - \frac{\partial}{\partial z}x^2yz\right)\mathbf{j} + \left(\frac{\partial}{\partial x}xy^2z - \frac{\partial}{\partial y}x^2yz\right)\mathbf{k}$$

$$= (xz^2 - xy^2)\mathbf{i} + (x^2y - yz^2)\mathbf{j} + (y^2z - x^2z)\mathbf{k}$$

$$\text{curl}\,\mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\cos y & xy^2 & 0 \end{vmatrix}$$

$$= \left[0 - \frac{\partial}{\partial z}xy^2\right]\mathbf{i} - \left[0 - \frac{\partial}{\partial z}(x\cos y)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}xy^2 - \frac{\partial}{\partial y}(x\cos y)\right]\mathbf{k}$$

$$= (y^2 + x\sin y)\mathbf{k}$$ ■

EXAMPLE 4 A constant vector field has divergence and curl zero

Let $\mathbf{F}$ be a constant vector field. Show that $\text{div}\,\mathbf{F} = 0$ and $\text{curl}\,\mathbf{F} = \mathbf{0}$.

Solution

Let $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ for constants a, b, and c. Then

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(a) + \frac{\partial}{\partial y}(b) + \frac{\partial}{\partial z}(c) = 0$$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a & b & c \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$ ■

WARNING Example 4 shows that the divergence and curl of a constant vector field are zero, but this does **not** mean that if $\text{div}\,\mathbf{F} = 0$ and $\text{curl}\,\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ must be a constant. For instance, the nonconstant vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$$

has both $\text{div}\,\mathbf{F} = 0$ and $\text{curl}\,\mathbf{F} = \mathbf{0}$.

TECHNOLOGY NOTE

Derive, *Maple*, MATLAB, and *Mathematica* will carry out most vector operations. For Example 2, we define $\mathbf{G}(x, y, z) = x\mathbf{i} + y^3z^2\mathbf{j} + xz^3\mathbf{k}$, and use one of these programs:

$\text{Div}[x, y^3z^2, xz^3]$ simplifies to $3xz^2 + 3y^2z^2 + 1$

You may need to enter a zero, as with $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^3\mathbf{j}$, as shown here:

$\text{Div}[x^2y, xy^3, 0]$ simplifies to $x(3y^2 + 2y)$

Finally, consider the vector field $\mathbf{F}$ from Example 3, $\mathbf{F} = x^2yz\mathbf{i} + xy^2z\mathbf{j} + xyz^2\mathbf{k}$. We can find the curl:

$\text{Curl}[x^2yz, xy^2z, xyz^2]$ simplifies to $[x(z^2 - y^2), y(x^2 - z^2), z(y^2 - x^2)]$

Combinations of the gradient, divergence, and curl appear in a variety of applications. In particular, note that if f is a differentiable scalar function, its gradient ∇f is a vector field, and we can compute

$$\begin{aligned}\operatorname{div}\nabla f &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\\ &= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\\ &= \nabla\cdot\nabla f\end{aligned}$$

In the following box, we introduce some special notation and terminology for this operation.

The Laplacian Operator

Let $f(x, y, z)$ define a function with continuous first and second partial derivatives. Then the **Laplacian of f** is

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} = f_{xx}+f_{yy}+f_{zz}$$

The equation $\nabla^2 f = 0$ is called **Laplace's equation**, and a function that satisfies such an equation in a region D is said to be **harmonic** in D.

The Laplacian is named for the French mathematician Pierre Laplace (1749–1827). See the Historical Quest, Section 12.3, Problem 62.

EXAMPLE 5 Showing a function is harmonic

Show that $f(x, y) = e^x \cos y$ is harmonic.

Solution

$$f_x(x, y) = e^x\cos y \quad\text{and}\quad f_{xx}(x, y) = e^x\cos y$$
$$f_y(x, y) = -e^x\sin y \quad\text{and}\quad f_{yy}(x, y) = -e^x\cos y$$

The Laplacian of f is given by

$$\begin{aligned}\nabla^2 f(x, y) &= f_{xx}(x, y) + f_{yy}(x, y) \qquad f_{zz} = 0\\ &= e^x\cos y - e^x\cos y = 0\end{aligned}$$

Thus, f is harmonic. ■

TECHNOLOGY NOTE

Derive, *Maple*, MATLAB, and *Mathematica* will find the Laplacian of a given function. For Example 5, we obtain

LAPLACIAN(e^x cos(y), [x, y]) which gives 0

If you have access to this technology, verify that

LAPLACIAN(x^2y^3z, [x, y, z]) yields $6x^2yz + 2y^3z$

In many ways, the study of electricity and magnetism is analogous to that of fluid dynamics, and the curl and divergence play an important role in this study. In electromagnetic theory, it is often convenient to regard interaction between electrical charges as forces somewhat like the gravitational force between masses and then to seek quantitative measure of these forces.

One of the great scientific achievements of the nineteenth century was the discovery of the laws of electromagnetism by the English scientist James Clerk Maxwell (see Historical Quest, Section 13.7, Problem 32). These laws have an elegant expression in terms of the divergence and curl. It is known empirically that the force acting on a charge due to an electromagnetic field depends on the position, velocity, and amount of the particular charge, and not on the number of other charges that may be present or how those other charges are moving. Suppose a charge is located at the point (x, y, z) at time t, and consider the electric intensity field $\mathbf{E}(x, y, z, t)$ and the magnetic intensity field $\mathbf{H}(x, y, z, t)$. Then the behavior of the resulting electromagnetic field is determined by

$$\operatorname{div}\mathbf{E} = \frac{Q}{\epsilon} \qquad \operatorname{div}(\mu\mathbf{H}) = 0$$

$$\operatorname{curl}\mathbf{E} = -\frac{\partial}{\partial t}(\mu\mathbf{H}) \qquad c^2(\operatorname{curl}\mathbf{B}) = \frac{\partial\mathbf{E}}{\partial t} + \frac{\mathbf{J}}{\epsilon}$$

where Q is the *electric charge density* (charge per unit volume), $\mathbf{J}$ is the *electric current density* (rate at which the charge flows through a unit area per second), $\mathbf{B}$ is the magnetic flux density, c is the speed of light, and μ and ϵ are constants called the *permeability* and *permittivity*, respectively. Working with these equations and terms is beyond the scope of this course, but if you are interested there are many references you can consult. One of the best (despite being almost 40 years old) is the classic *Feynman Lectures in Physics* (Reading, Mass.: Addison-Wesley, 1963), by Nobel laureate Richard Feynman, Robert Leighton, and Matthew Sands.

13.1 PROBLEM SET

1. **Exploration Problem** Discuss the del operator and its use in computing the divergence and curl.
2. **Exploration Problem** Discuss the difference between a vector valued function and a vector field.

In Problems 3–6, find div **F** *and* curl **F** *for the given vector function.*

3. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j} + z^3\mathbf{k}$
4. $\mathbf{F}(x, y) = \mathbf{i} + (x^2 + y^2)\mathbf{j}$
5. $\mathbf{F}(x, y, z) = 2y\mathbf{j}$
6. $\mathbf{F}(x, y, z) = z\mathbf{i} - \mathbf{j} + 2y\mathbf{k}$

In Problems 7–12, find div **F** *and* curl **F** *for each vector field* **F** *at the given point.*

7. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ at $(2, -1, 3)$
8. $\mathbf{F}(x, y, z) = xz\mathbf{i} + y^2z\mathbf{j} + xz\mathbf{k}$ at $(1, -1, 2)$
9. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ at $(1, 2, 3)$
10. $\mathbf{F}(x, y, z) = (\cos y)\mathbf{i} + (\sin y)\mathbf{j} + \mathbf{k}$ at $\left(\frac{\pi}{4}, \pi, 0\right)$
11. $\mathbf{F}(x, y, z) = e^{-xy}\mathbf{i} + e^{xz}\mathbf{j} + e^{yz}\mathbf{k}$ at $(3, 2, 0)$
12. $\mathbf{F}(x, y, z) = (e^{-x}\sin y)\mathbf{i} + (e^{-x}\cos y)\mathbf{j} + \mathbf{k}$ at $(1, 3, -2)$

Find div **F** *and* curl **F** *for each vector field* **F** *given in Problems 13–28.*

13. $\mathbf{F} = (\sin x)\mathbf{i} + (\cos y)\mathbf{j}$
14. $\mathbf{F} = (-\cos x)\mathbf{i} + (\sin y)\mathbf{j}$
15. $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$
16. $\mathbf{F} = -x\mathbf{i} + y\mathbf{j}$
17. $\mathbf{F} = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$
18. $\mathbf{F} = x^2\mathbf{i} - y^2\mathbf{j}$
19. $\mathbf{F} = ax\mathbf{i} + by\mathbf{j} + c\mathbf{k}$ for constants a, b, and c
20. $\mathbf{F} = (e^x \sin y)\mathbf{i} + (e^x \cos y)\mathbf{j} + \mathbf{k}$
21. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$
22. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$
23. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
24. $\mathbf{F} = 2xz\mathbf{i} + 2yz^2\mathbf{j} - \mathbf{k}$
25. $\mathbf{F} = xyz\mathbf{i} + x^2y^2z^2\mathbf{j} + y^2z^3\mathbf{k}$
26. $\mathbf{F} = (\ln z)\mathbf{i} + e^{xy}\mathbf{j} + \tan^{-1}\left(\dfrac{x}{z}\right)\mathbf{k}$
27. $\mathbf{F} = (z^2e^{-x})\mathbf{i} + (y^3 \ln z)\mathbf{j} + (xe^{-y})\mathbf{k}$
28. $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

Determine whether each scalar function in Problems 29–32 is harmonic.

29. $u(x, y, z) = e^{-x}(\cos y - \sin y)$
30. $v(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$
31. $w(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
32. $r(x, y, z) = xyz$
33. Show that the vector field $\mathbf{B} = y^2z\mathbf{i} + xz^3\mathbf{j} + y^2x^2\mathbf{k}$ is incompressible (that is, $\operatorname{div}\mathbf{B} = 0$).
34. Find div **F**, given that $\mathbf{F} = \nabla f$, where $f(x, y, z) = xy^3z^2$.
35. Find div **F**, given that $\mathbf{F} = \nabla f$, where $f(x, y, z) = x^2yz^3$.
36. If $\mathbf{F}(x, y, z) = 2\mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$, find $\operatorname{curl}(\mathbf{F} \times \mathbf{G})$.
37. If $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$, find $\operatorname{curl}(\mathbf{F} \times \mathbf{G})$.
38. If $\mathbf{F}(x, y, z) = 2\mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$, find $\operatorname{div}(\mathbf{F} \times \mathbf{G})$.
39. If $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$ and $\mathbf{G}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$, find $\operatorname{div}(\mathbf{F} \times \mathbf{G})$.
40. Let $\mathbf{A}$ be a constant vector and let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $\operatorname{div}(\mathbf{A} \times \mathbf{R}) = 0$.
41. Let $\mathbf{A}$ be a constant vector and let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $\operatorname{curl}(\mathbf{A} \times \mathbf{R}) = 2\mathbf{A}$.
42. If $\mathbf{F}(x, y) = u(x, y)\mathbf{i} + v(x, y)\mathbf{j}$, show that $\operatorname{curl}\mathbf{F} = \mathbf{0}$ if and only if $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$.

43. Consider a rigid body that is rotating about the z-axis (counterclockwise from above) with constant angular velocity $\omega = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. If P is a point in the body located at $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the velocity at P is given by the vector field $\mathbf{V} = \omega \times \mathbf{R}$.
   a. Express $\mathbf{V}$ in terms of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.
   b. Find div $\mathbf{V}$ and curl $\mathbf{V}$.

44. **Exploration Problem** If $\mathbf{F} = \langle f, g, h \rangle$ is an arbitrary vector field whose components are twice differentiable, what can be said about curl(curl $\mathbf{F}$)?

45. Which (if any) of the following is the same as div($\mathbf{F} \times \mathbf{G}$) for all vector fields $\mathbf{F}$ and $\mathbf{G}$?

   I. (div $\mathbf{F}$)(div $\mathbf{G}$)
   II. (curl $\mathbf{F}$) $\cdot\, \mathbf{G} - \mathbf{F} \cdot$ (curl $\mathbf{G}$)
   III. $\mathbf{F}$(div $\mathbf{G}$) + (div $\mathbf{F}$)$\mathbf{G}$
   IV. (curl $\mathbf{F}$) $\cdot\, \mathbf{G} + \mathbf{F} \cdot$ (curl $\mathbf{G}$)

In Problems 46–55, prove the given property for the vector fields $\mathbf{F}$ *and* $\mathbf{G}$, *scalar* c, *and scalar functions* f *and* g. *Assume that all required partial derivatives exist and are continuous.*

46. $\operatorname{div}(c\mathbf{F}) = c \operatorname{div} \mathbf{F}$
47. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$
48. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$
49. $\operatorname{curl}(c\mathbf{F}) = c \operatorname{curl} \mathbf{F}$
50. $\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + (\nabla f \times \mathbf{F})$
51. $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + (\nabla f \cdot \mathbf{F})$
52. $\operatorname{curl}(\nabla f + \operatorname{curl} \mathbf{F}) = \operatorname{curl}(\nabla f) + \operatorname{curl}(\operatorname{curl} \mathbf{F})$
53. $\operatorname{div}(f \nabla g) = f \operatorname{div} \nabla g + \nabla f \cdot \nabla g$
54. The curl of the gradient of a function is always $\mathbf{0}$. That is, $\nabla \times (\nabla f) = \mathbf{0}$.
55. The divergence of the curl of a vector field is 0. That is, div(curl $\mathbf{F}$) = 0.

In Problems 56–60, $\mathbf{R} = \langle x, y, z \rangle$, *and* $r = \|\mathbf{R}\| = \sqrt{x^2 + y^2 + z^2}$. *In each case, verify the given identity or answer the question.*

56. curl $\mathbf{R} = \mathbf{0}$; what is div $\mathbf{R}$?
57. $\operatorname{div}\left(\dfrac{1}{r^3}\mathbf{R}\right) = 0$
58. $\operatorname{curl}\left(\dfrac{1}{r^3}\mathbf{R}\right) = \mathbf{0}$
59. $\operatorname{div}(r\mathbf{R}) = 4r$
60. $\operatorname{div}(\nabla r) = \dfrac{2}{r}$
61. **Exploration Problem** State and prove an identity for $\operatorname{div}(\nabla(fg))$, where f and g are differentiable scalar functions of x, y, and z.
62. **Counterexample Problem** Let $\mathbf{F} = \langle x^2y, yz^2, zy^2 \rangle$. Either find a vector field $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl} \mathbf{G}$, or show that no such $\mathbf{G}$ exists.

13.2 Line Integrals

IN THIS SECTION definition of a line integral; line integrals with respect to x, y, and z; line integrals of vector fields; applications of line integrals: mass and work

A line integral is an integral whose integrand is evaluated at points along a curve in $\mathbb{R}^2$ or in $\mathbb{R}^3$. We will introduce line integrals in this section and show how they can be used for a variety of purposes in mathematics and physics.

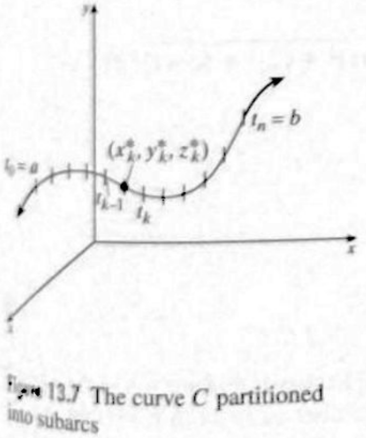


Figure 13.7 The curve C partitioned into subarcs

DEFINITION OF A LINE INTEGRAL

Let C be a smooth curve, with parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$ for $a \le t \le b$, that lies within the domain of a function $f(x, y, z)$. We say that C is **orientable** if it is possible to describe direction along the curve for increasing t.

To define a line integral, we begin by partitioning C into n subarcs, the kth of which has length Δs_k. Let (x_k^*, y_k^*, z_k^*) be a point chosen arbitrarily from the kth subarc (see Figure 13.7). Form the Riemann sum

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k$$

and let $\|\Delta s\|$ denote the largest subarc length in the partition. Then, if the limit

$$\lim_{\|\Delta s\| \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k$$

exists, we call this limit the *line integral* of f over C and denote it by $\displaystyle\int_C f(x, y, z)\, ds$.