63. Let $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and let $r = \|\mathbf{R}\| = \sqrt{x^2 + y^2 + z^2}$.
   a. Show that ∇r is a unit vector in the direction of $\mathbf{R}$.
   b. Show that $\nabla(r^n) = nr^{n-2}\mathbf{R}$, for any positive integer n.

64. Suppose the surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ both pass through the point $P_0(x_0, y_0, z_0)$ and that the gradients ∇F_0 and ∇G_0 both exist. Show that the two surfaces are tangent at P_0 if and only if $\nabla F_0 \times \nabla G_0 = \mathbf{0}$.

11.7 Extrema of Functions of Two Variables

IN THIS SECTION relative extrema, second partials test, absolute extrema of continuous functions, least squares approximation of data

There are many practical situations in which it is necessary or useful to know the largest and smallest values of a function of two variables. For example, if $T(x, y)$ is the temperature at a point (x, y) in a plate, where are the hottest and coldest points in the plate and what are these extreme temperatures? If a hazardous waste dump is bounded by the curve $F(x, y) = 0$, what are the largest and smallest distances to the boundary from a given interior point P_0? We begin our study of extrema with some terminology.

Absolute Extrema

The function $f(x, y)$ is said to have an **absolute maximum** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in the domain D of f. Similarly, f has an **absolute minimum** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in D. Collectively, absolute maxima and minima are called **absolute extrema**.

In Chapter 4, we located absolute extrema of a function of one variable by first finding *relative extrema*, those values of $f(x)$ that are larger or smaller than those at all nearby points. The relative extrema of a function of two variables may be defined as follows.

Relative Extrema

Let f be a function defined on a region containing (x_0, y_0). Then

$f(x_0, y_0)$ is a **relative maximum** if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0).

$f(x_0, y_0)$ is a **relative minimum** if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in an open disk containing (x_0, y_0).

Collectively, relative maxima and minima are called **relative extrema**.

RELATIVE EXTREMA

In Chapter 4, we observed that relative extrema of the function f correspond to "peaks and valleys" on the graph of f, and the same observation can be made about relative extrema in the two-variable case, as seen in Figure 11.37.

For a function f of one variable, we found that the relative extrema occur where $f'(x) = 0$ or $f'(x)$ does not exist. The following theorem shows that the relative extrema of a function of two variables can be located similarly.

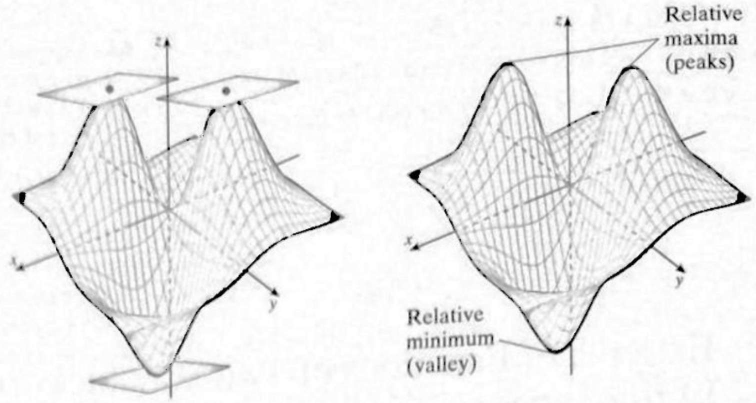


Figure 11.37 Relative extrema correspond to peaks and valleys

THEOREM 11.11 Partial derivative criteria for relative extrema

If f has a relative extremum (maximum or minimum) at $P_0(x_0, y_0)$ and partial derivatives f_x and f_y both exist at (x_0, y_0), then

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

Proof Let $F(x) = f(x, y_0)$. Then $F(x)$ must have a relative extremum at $x = x_0$, so $F'(x_0) = 0$, which means that $f_x(x_0, y_0) = 0$. Similarly, $G(y) = f(x_0, y)$ has a relative extremum at $y = y_0$, so $G'(y_0) = 0$ and $f_y(x_0, y_0) = 0$. Thus, we must have both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, as claimed. ❑

WARNING There is a horizontal tangent plane at each extreme point where the first partial derivatives exist. However, this **does not** say that whenever a horizontal tangent plane occurs at a point P, there must be an extremum there. All that can be said is that such a point P is a possible location for a relative extremum.

In single-variable calculus, we referred to a number x_0, where $f'(x_0)$ does not exist or $f'(x_0) = 0$ as a *critical number*. This terminology is extended to functions of two variables as follows.

Critical Points

A **critical point** of a function f defined on an open set D is a point (x_0, y_0) in D where either one of the following is true:

a. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.
b. At least one of $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist at (x_0, y_0).

EXAMPLE 1 Distinguishing critical points

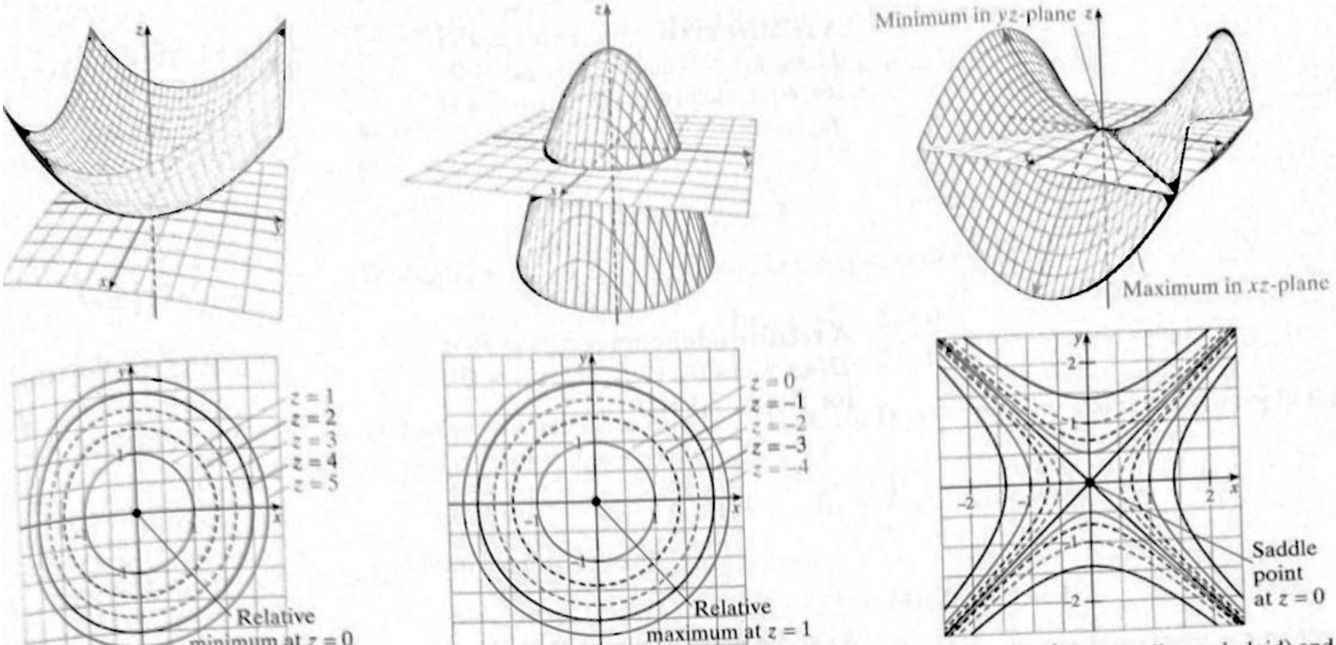
Discuss the nature of the critical point (0, 0) for the quadric surfaces

a. $z = x^2 + y^2$ **b.** $z = 1 - x^2 - y^2$ **c.** $z = y^2 - x^2$

Solution

The graphs of these quadric surfaces are shown in Figure 11.38. Let $f(x, y) = x^2+y^2$, $g(x, y) = 1 - x^2 - y^2$, and $h(x, y) = y^2 - x^2$. We find the critical points:

a. $f_x(x, y) = 2x$, $f_y(x, y) = 2y$; critical point (0, 0). The function f has a relative minimum at (0, 0) because x^2 and y^2 are both nonnegative, yielding $x^2 + y^2 > 0$ for all nonzero x and y.



a. $z = x^2 + y^2$ (paraboloid) and level curves; relative minimum at $(0, 0)$

b. $z = 1 - x^2 - y^2$ (paraboloid) and level curves; relative maximum at $(0, 0)$

c. $z = y^2 - x^2$ (hyperbolic paraboloid) and level curves; saddle point at $(0, 0)$

Figure 11.38 Classification of critical points

b. $g_x(x, y) = -2x$, $g_y(x, y) = -2y$; critical point $(0, 0)$. Since $z = 1 - x^2 - y^2$, it follows that $z \le 1$ with a relative maximum occurring where x^2 and y^2 are both 0; that is, at $(0, 0)$.

c. $h_x(x, y) = -2x$, $h_y(x, y) = 2y$; critical point $(0, 0)$. The function h has neither a relative maximum nor a relative minimum at $(0, 0)$. When $z = 0$, h is a minimum on the y-axis (where $x = 0$) and a maximum on the x-axis (where $y = 0$). ■

A critical point $P_0(x_0, y_0)$ is called a **saddle point** of $f(x, y)$ if every open disk centered at P_0 contains points in the domain of f that satisfy $f(x, y) > f(x_0, y_0)$ as well as points in the domain of f that satisfy $f(x, y) < f(x_0, y_0)$. An example of a saddle point is $(0, 0)$ on the hyperbolic paraboloid $z = y^2 - x^2$, as shown in Figure 11.38c.

SECOND PARTIALS TEST

The previous example points to the need for some sort of a test to determine the nature of a critical point. In Chapter 4, we developed the second derivative test for functions of one variable as a means for determining whether a particular critical number c of f corresponds to a relative maximum or minimum. If $f'(c) = 0$, then according to this test, a relative maximum occurs at $x = c$ if $f''(c) < 0$ and a relative minimum occurs if $f''(c) > 0$. If $f''(c) = 0$, the test is inconclusive. The analogous result for the two-variable case may be stated as follows.

THEOREM 11.12 **Second partials test**

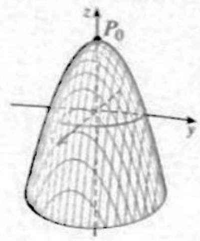
Let $f(x, y)$ have a critical point at $P_0(x_0, y_0)$ and assume that f has continuous second-order partial derivatives in a disk centered at (x_0, y_0). The *discriminant* of f is the expression

$$D = f_{xx} f_{yy} - f_{xy}^2$$

Then

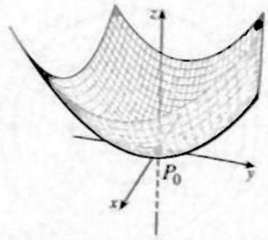
A **relative maximum** occurs at P_0 if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$ (or, equivalently, $D(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) < 0$).

$z = 1 - x^2 - y^2$ (See Ex 1b)



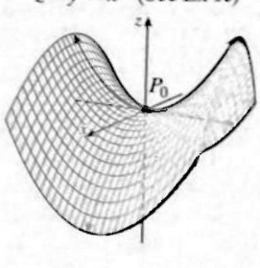
A **relative minimum** occurs at P_0 if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$ (or $f_{yy}(x_0, y_0) > 0$).

$z = x^2 + y^2$ (See Ex 1a)



A **saddle point** occurs at P_0 if $D(x_0, y_0) < 0$.

$z = y^2 - x^2$ (See Ex 1c)



Summary: For a critical point (a, b):

$D(a, b)$	$f_{xx}(a, b)$	**Type**
+	−	Rel. max.
+	+	Rel. min.
−	NA	Saddle point
0	NA	Inconclusive

If $D(x_0, y_0) = 0$, then the test is **inconclusive**. Nothing can be said about the nature of the surface at (x_0, y_0) without further analysis.

Proof The second partials test can be proved by using an extension of the Taylor series expansion (Section 8.8) that applies to functions of two variables $f(x, y)$. Details can be found in most advanced calculus texts (also see Problem 60). ❑

The discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ may be easier to remember in the equivalent determinant form

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

SMH The definition of a 2×2 determinant is presented in Section 2.8 of the *Student Mathematics Handbook*.

Note that if $D > 0$ at the critical point $P_0(x_0, y_0)$, then f_{xx} and f_{yy} must have the same sign. This is the reason that when $D > 0$, either $f_{xx} > 0$ or $f_{yy} > 0$ is enough to guarantee that a relative minimum occurs at P_0 (or a relative maximum if $f_{xx} < 0$ or $f_{yy} < 0$).

Geometrically, if $D > 0$ and $f_{xx} > 0$ and $f_{yy} > 0$ at P_0, then the surface $z = f(x, y)$ curves upward in all directions from the point $Q(x_0, y_0, z_0)$, so there is a relative minimum at Q. Likewise, if $D > 0$ and $f_{xx} < 0$ and $f_{yy} < 0$ at P_0, then the surface curves downward in all directions from P_0, which must therefore be a relative maximum. However, if $D < 0$ at P_0, the surface curves up from Q in some directions, and down in others, so Q must be a saddle point.

WARNING The second partials test says nothing about the geometry of the surface at Q if $D = 0$ at P_0. Example 4 and Problem 54 show that a relative minimum, a relative maximum, a saddle point, or something entirely different may occur if $D = 0$.

EXAMPLE 2 Using the second partials test to classify critical points

Find all relative extrema and saddle points of the function

$$f(x, y) = 2x^2 + 2xy + y^2 - 2x - 2y + 5$$

Solution

First, find the critical points:

$$f_x = 4x + 2y - 2 \qquad f_y = 2x + 2y - 2$$

Setting $f_x = 0$ and $f_y = 0$, we obtain the system of equations

$$\begin{cases} 4x + 2y - 2 = 0 \\ 2x + 2y - 2 = 0 \end{cases}$$

and solve to obtain $x = 0$, $y = 1$. Thus, $(0, 1)$ is the only critical point. To apply the second partials test, we obtain

$$f_{xx} = 4 \qquad f_{yy} = 2 \qquad f_{xy} = 2$$

and form the discriminant

$$D = f_{xx}f_{yy} - f_{xy}^2 = (4)(2) - 2^2 = 4$$

For the critical point $(0, 1)$ we have $D = 4 > 0$ and $f_{xx} = 4 > 0$, so there is a relative minimum at $(0, 1)$. ■

EXAMPLE 3 Second partials test with a relative minimum and a saddle point

Find all critical points on the graph of $f(x, y) = 8x^3 - 24xy + y^3$, and use the second partials test to classify each point as a relative extremum or a saddle point.

Solution

$$f_x(x, y) = 24x^2 - 24y, \qquad f_y(x, y) = -24x + 3y^2$$

To find the critical points, solve

$$\begin{cases} 24x^2 - 24y = 0 \\ -24x + 3y^2 = 0 \end{cases}$$

From the first equation, $y = x^2$; substitute this into the second equation to find

$$-24x + 3(x^2)^2 = 0$$

$$x(x^3 - 8) = 0$$

$$x(x-2)(x^2 + 2x + 4) = 0$$

$$x = 0, 2 \qquad \text{The solutions of } x^2 + 2x + 4 = 0 \text{ are not real.}$$

If $x = 0$, then $y = 0$, and if $x = 2$, then $y = 4$, so the critical points are $(0, 0)$ and $(2, 4)$. To obtain D, we first find $f_{xx}(x, y) = 48x$, $f_{xy}(x, y) = -24$, and $f_{yy}(x, y) = 6y$ to find $D(x, y) = (48x)(6y) - (-24)^2$ and then compute:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 48x & -24 \\ -24 & 6y \end{vmatrix} = 288xy - 576$$

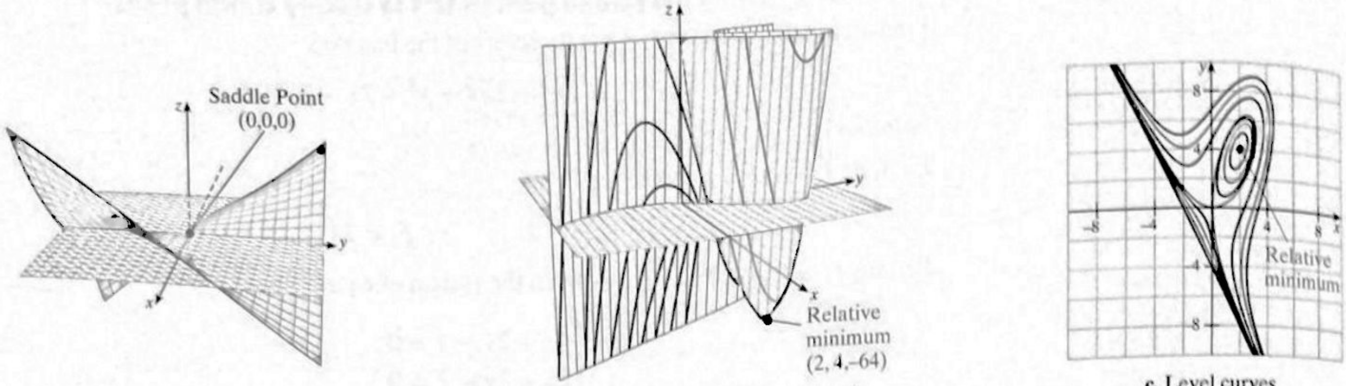
Critical point

(a, b)	$D(a, b)$	$f_{xx}(a, b)$	type
(0, 0)	Neg.		Saddle
(2, 4)	Pos.	Pos.	Rel. min.

At $(0, 0)$, $D = -576 < 0$, so there is a saddle point at $(0, 0)$.

At $(2, 4)$, $D = 288(2)(4) - 576 = 1{,}728 > 0$ and $f_{xx}(2, 4) = 96 > 0$, so there is a relative minimum at $(2, 4)$.

To view the situation graphically, we calculate the coordinates of the saddle point $(0, 0, 0)$, and the relative minimum $(2, 4, -64)$, as shown in Figure 11.39.



a. View near the origin (showing the saddle point)

b. View away from the origin (showing the relative minimum point)

c. Level curves

Figure 11.39 Graph of $f(x, y) = 8x^3 - 24xy + y^3$

EXAMPLE 4 Extrema when the second partials test fails

Find all relative extrema and saddle points on the graph of

$$f(x, y) = x^2y^4$$

The graph is shown in Figure 11.40.

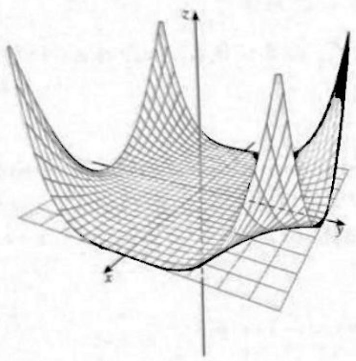


Figure 11.40 Graph of $f(x, y) = x^2y^4$

Solution

Since $f_x(x, y) = 2xy^4$, $f_y(x, y) = 4x^2y^3$, we see that the critical points occur only when $x = 0$ or $y = 0$; that is, every point on the x-axis or y-axis is a critical point. Because

$$f_{xx}(x, y) = 2y^4, \qquad f_{xy}(x, y) = 8xy^3, \qquad f_{yy}(x, y) = 12x^2y^2$$

the discriminant is

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2y^4 & 8xy^3 \\ 8xy^3 & 12x^2y^2 \end{vmatrix} = 24x^2y^6 - 64x^2y^6 = -40x^2y^6$$

Since $D = 0$ for any critical point $(x_0, 0)$ or $(0, y_0)$, the second partials test fails. However, $f(x, y) = 0$ for every critical point (because either $x = 0$ or $y = 0$, or both), and because $f(x, y) = x^2y^4 > 0$ when $x \neq 0$ and $y \neq 0$, it follows that each critical point must be a relative minimum. ■

ABSOLUTE EXTREMA OF CONTINUOUS FUNCTIONS

The extreme value theorem (Theorem 4.1) says that a function of a single variable f must attain both an absolute maximum and an absolute minimum on any closed, bounded interval $[a, b]$ on which it is continuous. In $\mathbb{R}^2$, a nonempty set S is **closed** if it contains its boundary (see the introduction to Section 11.2) and is **bounded** if it is contained in a disk. The extreme value theorem can be extended to functions of two variables in the following form.

THEOREM 11.13 Extreme value theorem for a function of two variables

A function of two variables $f(x, y)$ attains both an absolute maximum and an absolute minimum on any closed, bounded set S where it is continuous.

Proof The proof is found in most advanced calculus texts. ❑

To find the absolute extrema of a continuous function $f(x, y)$ on a closed, bounded set S, we proceed as follows:

Procedure for Determining Absolute Extrema

Given a function f that is continuous on a closed, bounded set S,

Step 1. Find all critical points of f in S.
Step 2. Find all points on the boundary of S where absolute extrema can occur.
Step 3. Compute the value of $f(x_0, y_0)$ for each of the points (x_0, y_0) found in steps 1 and 2.

Evaluation: The absolute maximum of f on S is the largest of the values computed in step 3, and the absolute minimum is the smallest of the computed values.

EXAMPLE 5 Finding absolute extrema

Find the absolute extrema of the function $f(x, y) = e^{x^2-y^2}$ over the disk $x^2 + y^2 \le 1$. The graph is shown in Figure 11.41.

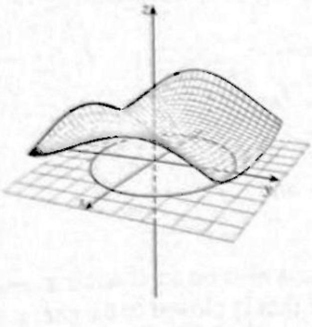


Figure 11.41 Graph of $f(x, y) = e^{x^2-y^2}$ over the disk $x^2+y^2 \le 1$

Solution

Step 1. $f_x(x, y) = 2xe^{x^2-y^2}$ and $f_y(x, y) = -2ye^{x^2-y^2}$. These partial derivatives are defined for all (x, y). Because $f_x(x, y) = f_y(x, y) = 0$ only when $x = 0$ and $y = 0$, it follows that $(0, 0)$ is the only critical point of f and it is inside the disk.

Step 2. Examine the values of f on the boundary curve $x^2 + y^2 = 1$. Because $y^2 = 1 - x^2$ on the boundary of the disk, we find that

$$f(x, y) = e^{x^2-(1-x^2)} = e^{2x^2-1}$$

We need to find the largest and smallest values of $F(x) = e^{2x^2-1}$ for $-1 \le x \le 1$. Since

$$F'(x) = 4xe^{2x^2-1}$$

we see that $F'(x) = 0$ only when $x = 0$ (since e^{2x^2-1} is always positive). At $x = 0$, we have $y^2 = 1 - 0^2$, so $y = \pm 1$; thus $(0, 1)$ and $(0, -1)$ are boundary critical points. At the endpoints of the interval $-1 \le x \le 1$, the corresponding points are $(1, 0)$ and $(-1, 0)$.

Step 3. Compute the value of f for the points found in steps 1 and 2:

Points to check	Compute $f(x_0, y_0) = e^{x_0^2-y_0^2}$
$(0, 0)$	$f(0, 0) = e^0 = 1$
$(0, 1)$	$f(0, 1) = e^{-1}$; minimum
$(0, -1)$	$f(0, -1) = e^{-1}$; minimum
$(1, 0)$	$f(1, 0) = e$; maximum
$(-1, 0)$	$f(-1, 0) = e$; maximum

Evaluation: As indicated in the preceding table, the absolute maximum value of f on the given disk is e, which occurs at $(1, 0)$ and $(-1, 0)$, and the absolute minimum value is e^{-1}, which occurs at $(0, 1)$ and $(0, -1)$. ■

In general, it can be difficult to show that a relative extremum is actually an absolute extremum. In practice, however, it is often possible to make the determination using physical or geometric considerations. Consider the following example.

EXAMPLE 6 Minimum distance from a point to a plane

Find the point on the plane $x + 2y + z = 5$ that is closest to the point $P(0, 3, 4)$.

Solution

If $Q(x, y, z)$ is a point on the plane $x + 2y + z = 5$, then $z = 5 - x - 2y$ and the distance from P to Q is

$$d = \sqrt{(x-0)^2 + (y-3)^2 + (z-4)^2}$$
$$= \sqrt{x^2 + (y-3)^2 + (5-x-2y-4)^2}$$

Instead of minimizing d, we minimize

$$f(x, y) = d^2 = x^2 + (y-3)^2 + (1-x-2y)^2$$

since the minimum of d will occur at the same points where d^2 is also minimized.

To minimize $f(x, y)$, we first determine the critical points of f by solving the system

$$f_x = 2x - 2(1-x-2y) = 4x + 4y - 2 = 0$$
$$f_y = 2(y-3) - 4(1-x-2y) = 4x + 10y - 10 = 0$$

We obtain $x = -\frac{5}{6}$, $y = \frac{4}{3}$, and since

$$f_{xx} = 4, \qquad f_{yy} = 10, \qquad f_{xy} = 4$$

we find that

$$D = f_{xx}f_{yy} - f_{xy}^2 = 4(10) - 4^2 > 0 \quad \text{and} \quad f_{xx} = 4 > 0$$

so a relative minimum occurs at $\left(-\frac{5}{6}, \frac{4}{3}\right)$.

Intuitively, we see that this relative minimum must also be an absolute minimum because there must be exactly one point on the plane that is closest to the given point. The corresponding z-value is $z = 5 - \left(-\frac{5}{6}\right) - 2\left(\frac{4}{3}\right) = \frac{19}{6}$. Thus, the closest point on the plane is $Q\left(-\frac{5}{6}, \frac{4}{3}, \frac{19}{6}\right)$, and the minimum distance is

$$d = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{4}{3} - 3\right)^2 + \left[1 + \frac{5}{6} - 2\left(\frac{4}{3}\right)\right]^2} = \sqrt{\frac{25}{6}} = \frac{5}{\sqrt{6}}$$

Check: You might want to check your work by using the formula for the distance from a point to a plane in $\mathbb{R}^3$ (Theorem 9.9):

$$d = \left|\frac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}\right| = \left|\frac{0 + 2(3) + 4 - 5}{\sqrt{1^2 + 2^2 + 1^2 33}}\right| = \frac{5}{\sqrt{6}}$$ ■

LEAST SQUARES APPROXIMATION OF DATA

In the following example, calculus is applied to justify a formula used in statistics and in many applications in the social and physical sciences.

EXAMPLE 7 Least squares approximation of data

Suppose data consisting of n points $P_1, \ldots, P_n$ are known, and we wish to find a function $y = f(x)$ that fits the data reasonably well. In particular, suppose we wish to find a line $y = mx + b$ that "best fits" the data in the sense that the sum of the squares of the vertical distances from each data point to the line is minimized.

Solution

We wish to find values of m and b that minimize the sum of the squares of the differences between the y-values and the line $y = mx + b$. The line that we seek is called the **regression line**. Suppose that the point P_k has components (x_k, y_k). At this point the value on the regression line is $y = mx_k + b$ and the value of the data point is y_k. The "error" caused by using the point on the regression line rather than the actual data point can be measured by the difference

$$y_k - (mx_k + b)$$

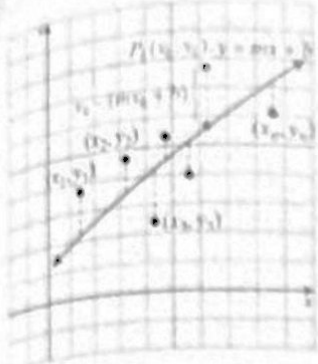


Figure 11.42 Least squares approximation of data

The data points may be above the regression line for some values of k and below the regression line for other values of k. We see that we need to minimize the function that represents the sum of the *squares* of all these differences:

$$F(m,b)=\sum_{k=1}^{n}[y_k-(mx_k+b)]^2$$

The situation is illustrated in Figure 11.42. To find where F is minimized, we first compute the partial derivatives.

$$\begin{aligned}
F_m(m,b)&=\sum_{k=1}^{n}2[y_k-(mx_k+b)](-x_k)\\
&=2m\sum_{k=1}^{n}x_k^2+2b\sum_{k=1}^{n}x_k-2\sum_{k=1}^{n}x_ky_k\\
F_b(m,b)&=\sum_{k=1}^{n}2[y_k-(mx_k+b)](-1)\\
&=2m\sum_{k=1}^{n}x_k+2b\sum_{k=1}^{n}1-2\sum_{k=1}^{n}y_k\\
&=2m\sum_{k=1}^{n}x_k+2bn-2\sum_{k=1}^{n}y_k
\end{aligned}$$

Set each of these partial derivatives equal to 0 to find the critical values (see Problem 61).

$$m=\frac{n\displaystyle\sum_{k=1}^{n}x_ky_k-\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}y_k\right)}{n\displaystyle\sum_{k=1}^{n}x_k^2-\left(\sum_{k=1}^{n}x_k\right)^2}\quad\text{and}\quad b=\frac{\displaystyle\sum_{k=1}^{n}x_k^2\sum_{k=1}^{n}y_k-\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}x_ky_k\right)}{n\displaystyle\sum_{k=1}^{n}x_k^2-\left(\sum_{k=1}^{n}x_k\right)^2}$$

It can be shown that these values of m and b yield an absolute minimum for $F(m,b)$. ■

Most applications of the **least squares formula** stated in Example 7 involve using a calculator or computer software. The following technology note provides an example.

TECHNOLOGY NOTE

Many calculators will carry out the calculations required by the least squares approximation procedure. Look at your owner's manual for specifics. Most calculators allow you to input data with keys labeled STAT and DATA. After the data are input, the m and b values are given by pressing the LinReg choice. For example, ten people are given a standard IQ test. Their scores are then compared with their high school grades:

IQ:	117	105	111	96	135	81	103	99	107	109
GPA:	3.1	2.8	2.5	2.8	3.4	1.9	2.1	3.2	2.9	2.3

A calculator output shows: $m=.0224144711$ and $b=.3173417224$. A scatter diagram with the least squares line is shown in Figure 11.43.

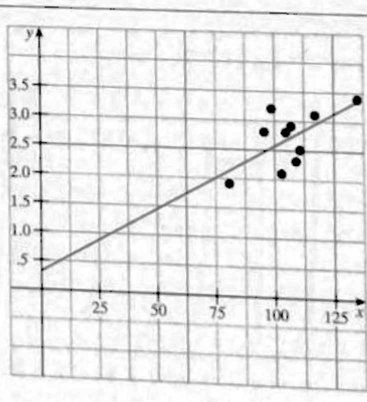


Figure 11.43 Scatter diagram and least squares line

11.7 PROBLEM SET

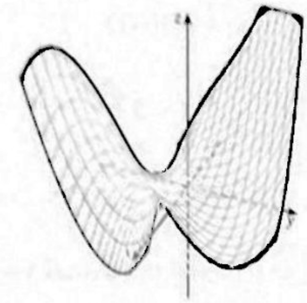
1. **WHAT DOES THIS SAY?** Describe what is meant by a critical point.

2. **WHAT DOES THIS SAY?** Describe a procedure for classifying relative extrema.

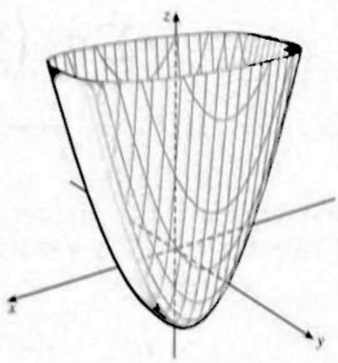
3. **WHAT DOES THIS SAY?** Describe a procedure for determining absolute extrema on a closed, bounded set S.

Find the critical points in Problems 4–23, and classify each point as a relative maximum, a relative minimum, or a saddle point.

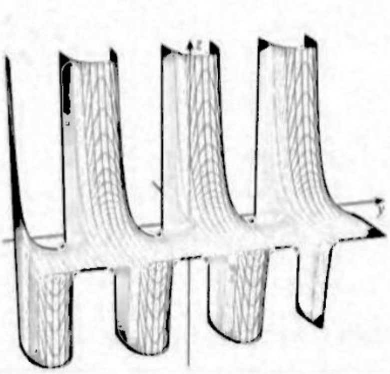
4. $f(x, y) = 2x^2 - 4xy + y^3 + 2$



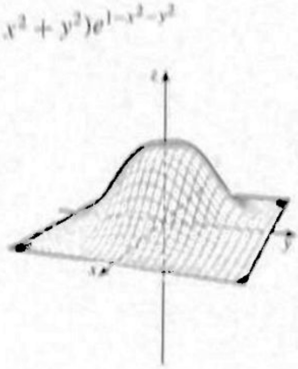
5. $f(x, y) = (x - 2)^2 + (y - 3)^4$



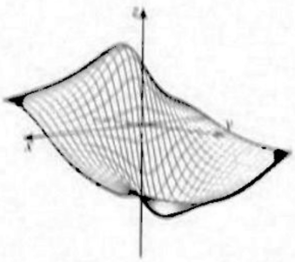
6. $f(x, y) = e^{-x} \sin y$



7. $f(x, y) = (1 + x^2 + y^2)e^{1-x^2-y^2}$



8. $f(x, y) = \dfrac{9x}{x^2 + y^2 + 1}$



9. $f(x, y) = x^2 + xy + y^2$

10. $f(x, y) = xy - x + y$

11. $f(x, y) = -x^3 + 9x - 4y^2$

12. $f(x, y) = e^{-(x^2+y^2)}$

13. $f(x, y) = (x^2 + 2y^2)e^{1-x^2-y^2}$

14. $f(x, y) = e^{xy}$

15. $f(x, y) = x^{-1} + y^{-1} + 2xy$

16. $f(x, y) = (x - 4)\ln(xy)$

17. $f(x, y) = x^3 + y^3 + 3x^2 - 18y^2 + 81y + 5$

18. $f(x, y) = 2x^3 + y^3 + 3x^2 - 3y - 12x - 4$

19. $f(x, y) = x^2 + y^2 - 6xy + 9x + 5y + 2$

20. $f(x, y) = x^2 + y^2 + \dfrac{32}{xy}$

21. $f(x, y) = x^2 + y^3 + \dfrac{768}{x + y}$

22. $f(x, y) = 3xy^2 - 2x^2y + 36xy$

23. $f(x, y) = 3x^2 + 12x + 8y^3 - 12y^2 + 7$

Find the absolute extrema of f on the closed bounded set S in the plane as described in Problems 24–30.

24. $f(x, y) = 2x^2 - y^2$; S is the disk $x^2 + y^2 \le 1$.

25. $f(x, y) = xy - 2x - 5y$; S is the triangular region with vertices $(0, 0)$, $(7, 0)$, and $(7, 7)$.

26. $f(x, y) = x^2 + 3y^2 - 4x + 2y - 3$; S is the square region with vertices $(0, 0)$, $(3, 0)$, $(3, -3)$, and $(0, -3)$.

27. $f(x, y) = 2\sin x + 5\cos y$; S is the rectangular region with vertices $(0, 0)$, $(2, 0)$, $(2, 5)$, and $(0, 5)$.

28. $f(x, y) = e^{x^2+2x+y^2}$; S is the disk $x^2 + 2x + y^2 \le 0$.

29. $f(x, y) = x^2 + xy + y^2$; S is the disk $x^2 + y^2 \le 1$.

30. $f(x, y) = x^2 - 4xy + y^3 + 4y$; S is the square region $0 \le x \le 2$, $0 \le y \le 2$.

Find the least squares regression line for the data points given in Problems 31–34.

31. $(-2,-3)$, $(-1,-1)$, $(0,1)$, $(1,3)$, $(3,5)$
32. $(0,1)$, $(1,1.6)$, $(2.2,3)$, $(3.1,3.9)$, $(4,5)$
33. $(3,5.72)$, $(4,5.31)$, $(6.2,5.12)$, $(7.52,5.32)$, $(8.03,5.67)$
34. $(-4,2)$, $(-3,1)$, $(0,0)$, $(1,-3)$, $(2,-1)$, $(3,-2)$
35. Find all points on the surface $y^2 = 4 + xz$ that are closest to the origin.
36. Find all points in the plane $x + 2y + 3z = 4$ in the first octant where $f(x,y,z) = x^2yz^3$ has a maximum value.
37. A rectangular box with no top is to have a fixed volume. What should its dimensions be if we want to use the least amount of material in its construction?
38. A wire of length L is cut into three pieces that are bent to form a circle, a square, and an equilateral triangle. How should the cuts be made to minimize the sum of the total area?
39. Find three positive numbers whose sum is 54 and whose product is as large as possible.
40. A dairy produces whole milk and skim milk in quantities x and y pints, respectively. Suppose the price (in cents) of whole milk is $p(x) = 100 - x$ and that of skim milk is $q(y) = 100 - y$, and also assume that $C(x,y) = x^2 + xy + y^2$ is the joint-cost function of the commodities. Maximize the profit
$$P(x,y) = px + qy - C(x,y)$$
41. Let R be the triangular region in the xy-plane with vertices $(-1,-2)$, $(-1,2)$, and $(3,2)$. A plate in the shape of R is heated so that the temperature at (x,y) is
$$T(x,y) = 2x^2 - xy + y^2 - 2y + 1$$
(in degrees Celsius). At what point in R or on its boundary is T maximized? Where is T minimized? What are the extreme temperatures?
42. A particle of mass m in a rectangular box with dimensions x, y, z has *ground state energy*
$$E(x,y,z) = \frac{k^2}{8m}\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$$
where k is a physical constant. If the volume of the box is fixed (say $V_0 = xyz$), find the values of x, y, and z that minimize the ground state energy.
43. A manufacturer produces two different kinds of graphing calculators, A and B, in quantities x and y (units of 1,000), respectively. If the revenue function (in dollars) is $R(x,y) = -x^2 - 2y^2 + 2xy + 8x + 5y$, find the quantities of A and B that should be produced to maximize revenue.
44. Suppose we wish to construct a closed rectangular box with volume 32 ft^3. Three different materials will be used in the construction. The material for the sides costs \$1 per square foot, the material for the bottom costs \$3 per square foot, and the material for the top costs \$5 per square foot. What are the dimensions of the least expensive such box?
45. **Modeling Problem** A store carries two competing brands of bottled water, one from California and the other from upstate New York. To model this situation, assume the owner of the store can obtain both at a cost of \$2/bottle. Also assume that if the California water is sold for x dollars per bottle and the New York water for y dollars per bottle, then consumers will buy approximately $40 - 50x + 40y$ bottles of California water and $20 + 60x - 70y$ bottles of the New York water each day. How should the owner price the bottled water to generate the largest possible profit?
46. **Modeling Problem** A telephone company is planning to introduce two new types of executive communications systems that it hopes to sell to its largest commercial customers. To create a model to determine the maximum profit, it is assumed that if the first type of system is priced at x hundred dollars per system and the second type at y hundred dollars per system, approximately $40 - 8x + 5y$ consumers will buy the first type and $50 + 9x - 7y$ will buy the second type. If the cost of manufacturing the first type is \$1,000 per system and the cost of manufacturing the second type is \$3,000 per system, how should the telephone company price the systems to generate maximum profit?
47. **Modeling Problem** A manufacturer with exclusive rights to a sophisticated new industrial machine is planning to sell a limited number of the machines to both foreign and domestic firms. The price the manufacturer can expect to receive for the machines will depend on the number of machines made available. For example, if only a few of the machines are placed on the market, competitive bidding among prospective purchasers will tend to drive the price up. It is estimated that if the manufacturer supplies x machines to the domestic market and y machines to the foreign market, the machines will sell for $60 - 0.2x + 0.05y$ thousand dollars apiece at home and $50 - 0.1y + 0.05x$ thousand dollars apiece abroad. If the manufacturer can produce the machines at a total cost of \$10,000 apiece, how many should be supplied to each market to generate the largest possible profit?
48. A college admissions officer, Dr. Westfall, has compiled the following data relating students' high-school and college GPAs:

HS GPA	2.0	2.5	3.0	3.0	3.5	3.5	4.0	4.0
College GPA	1.5	2.0	2.5	3.5	2.5	3.0	3.0	3.5

Plot the data points on a graph and find the equation of the regression line for these data. Then use the regression line to predict the college GPA of a student whose high school GPA is 3.75.

49. It is known that if an ideal spring is displaced a distance y from its natural length by a force (weight) x, then $y = kx$, where k is the so-called spring constant. To compute this constant for a particular spring, a scientist obtains the following data:

x (lb)	5.2	7.3	8.4	10.12	12.37
y (in.)	11.32	15.56	17.44	21.96	26.17

Based on these data, what is the "best" choice for k?

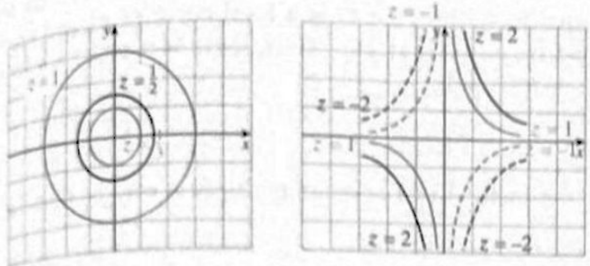
50. **Exploration Problem** The following table gives the approximate U.S. census figures (in millions):

Year:	1900	1910	1920	1930	1940
Population:	76.2	92.2	106.0	123.2	132.1

Year:	1950	1960	1970	1980	1990
Population:	151.3	179.3	203.3	226.5	248.7

58. **Exploration Problem** Consider the function $f(x, y) = (y - x^2)(y - 2x^2)$. Discuss the behavior of this function at $(0, 0)$.

59. **Exploration Problem** Sometimes the critical points of a function can be classified by looking at the level curves. In each case shown in Figure 11.45, determine the nature of the critical point(s) of $z = f(x, y)$ at $(0, 0)$.



a. b.

Figure 11.45 Problem 59

60. Prove the second partials test. *Hint:* Compute the second directional derivative of f in the direction of the unit vector $\mathbf{u} = h\mathbf{i} + k\mathbf{j}$ and complete the square.

61. Verify the formulas for m and b associated with the least squares approximation.

62. This problem involves a generalization of the least squares procedure, in which a "least squares plane" is found to produce the best fit for a given set of data. A researcher knows that the quantity z is related to x and y by a formula of the form $z = k_1x + k_2y$, where k_1 and k_2 are physical constants. To determine these constants, she conducts a series of experiments, the results of which are tabulated as follows:

x	1.20	0.86	1.03	1.65	−0.95	−1.07
y	0.43	1.92	1.52	−1.03	1.22	−0.06
z	3.21	5.73	2.22	0.92	−1.11	−0.97

Modify the method of least squares to find a "best approximation" for k_1 and k_2.

11.8 Lagrange Multipliers

IN THIS SECTION method of Lagrange multipliers, constrained optimization problems, Lagrange multipliers with two parameters, a geometric interpretation of Lagrange's theorem

METHOD OF LAGRANGE MULTIPLIERS

In many applied problems, a function of two variables is to be optimized subject to a restriction or **constraint** on the variables. For example, consider a container heated in such a way that the temperature at the point (x, y, z) in the container is given by the function $T(x, y, z)$. Suppose that the surface $z = f(x, y)$ lies in the container, and that we wish to find the point on $z = f(x, y)$ where the temperature is the greatest. In other words, *What is the maximum value of T subject to the constraint $z = f(x, y)$, and where does this maximum value occur?*

THEOREM 11.14 Lagrange's theorem

Assume that f and g have continuous first partial derivatives and that f has an extremum at $P_0(x_0, y_0)$ on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, there is a number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Proof Denote the constraint curve $g(x, y) = c$ by C, and note that C is smooth. We represent this curve by the vector function

$$\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

for all t in an open interval I, including t_0 corresponding to P_0, where $x'(t)$ and $y'(t)$ exist and are continuous. Let $F(t) = f(x(t), y(t))$ for all t in I, and apply the chain rule to obtain

$$F'(t) = f_x(x(t), y(t))\frac{dx}{dt} + f_y(x(t), y(t))\frac{dy}{dt} = \nabla f(x(t), y(t)) \cdot \mathbf{R}'(t)$$

Because $f(x, y)$ has an extremum at P_0, we know that $F(t)$ has an extremum at t_0, the value of t that corresponds to P_0 (that is, P_0 is the point on C where $t = t_0$). Therefore, we have $F'(t_0) = 0$ and

$$F'(t_0) = \nabla f(x(t_0), y(t_0)) \cdot \mathbf{R}'(t_0) = 0$$

If $\nabla f(x(t_0), y(t_0)) = 0$, then $\lambda = 0$, and the condition $\nabla f = \lambda \nabla g$ is satisfied trivially. If $\nabla f(x(t_0), y(t_0)) \neq 0$, then $\nabla f(x(t_0), y(t_0))$ is orthogonal to $\mathbf{R}'(t_0)$. Because $\mathbf{R}'(t_0)$ is tangent to the constraint curve C, it follows that $\nabla f(x_0, y_0)$ is normal to C. But $\nabla g(x_0, y_0)$ is also normal to C (because C is a level curve of g), and we conclude that ∇f and ∇g must be *parallel* at P_0. Thus, there is a scalar λ such that $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$, as required. ☐

CONSTRAINED OPTIMIZATION PROBLEMS

The general procedure for the method of Lagrange multipliers may be described as follows.

Procedure for the Method of Lagrange Multipliers

Suppose f and g satisfy the hypotheses of Lagrange's theorem, and that $f(x, y)$ has an extremum subject to the constraint $g(x, y) = c$. Then to find the extreme value, proceed as follows:

1. Simultaneously solve the following three equations for x, y, and λ:

$$f_x(x, y) = \lambda g_x(x, y) \qquad f_y(x, y) = \lambda g_y(x, y) \qquad g(x, y) = c$$

2. Evaluate f at all points found in step 1 and all points on the boundary of the constraint. The extremum we seek must be among these values.

EXAMPLE 1 Optimization with Lagrange multipliers

Given that the largest and smallest values of $f(x, y) = 1 - x^2 - y^2$ subject to the constraint $x + y = 1$ with $x \geq 0$, $y \geq 0$ exist, use the method of Lagrange multipliers to find these extrema.

Solution

Because the constraint is $x + y = 1$, let $g(x, y) = x + y$

$$f_x(x, y) = -2x \qquad f_y(x, y) = -2y \qquad g_x(x, y) = 1 \qquad g_y(x, y) = 1$$

Form the system

$$\begin{cases} -2x = \lambda(1) & \leftarrow f_x(x, y) = \lambda g_x(x, y) \\ -2y = \lambda(1) & \leftarrow f_y(x, y) = \lambda g_y(x, y) \\ x + y = 1 & \leftarrow g(x, y) = 1 \end{cases}$$

SMH This system is solved as Example 3.4 in Section 3.1 of the *Student Mathematics Handbook.*

The only solution is $x = \frac{1}{2}$, $y = \frac{1}{2}$.

$$f\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 1 - \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2}$$

The endpoints of the line segment

$$x + y = 1 \qquad \text{for } x \geq 0, y \geq 0$$

are at $(1, 0)$ and $(0, 1)$, and we find that

$$f(1, 0) = 1 - 1^2 - 0^2 = 0$$
$$f(0, 1) = 1 - 0^2 - 1^2 = 0$$

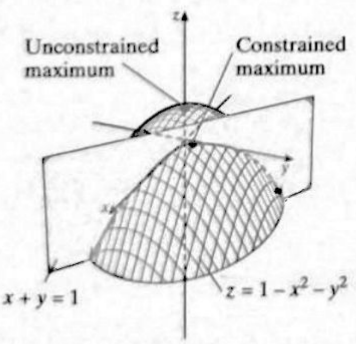


Figure 11.46 The maximum is the high point of the curve of intersection of the surface and the plane.

Therefore, the maximum value is $\frac{1}{2}$ at $\left(\frac{1}{2}, \frac{1}{2}\right)$, and the minimum value is 0 at $(1, 0)$ and $(0, 1)$. (See Figure 11.46.) ■

The method of Lagrange multipliers extends naturally to functions of three or more variables. If a function $f(x, y, z)$ has an extreme value subject to a constraint $g(x, y, z) = c$, then the extremum occurs at a point (x_0, y_0, z_0) such that $g(x_0, y_0, z_0) = c$ and $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$ for some number λ. Here is an example.

EXAMPLE 2 Hottest and coldest points on a plate

A container in $\mathbb{R}^3$ has the shape of the cube given by $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$. A plate is placed in the container in such a way that it occupies that portion of the plane $x + y + z = 1$ that lies in the cubical container. If the container is heated so that the temperature at each point (x, y, z) is given by

$$T(x, y, z) = 4 - 2x^2 - y^2 - z^2$$

in hundreds of degrees Celsius, what are the hottest and coldest points on the plate? You may assume these extreme temperatures exist.

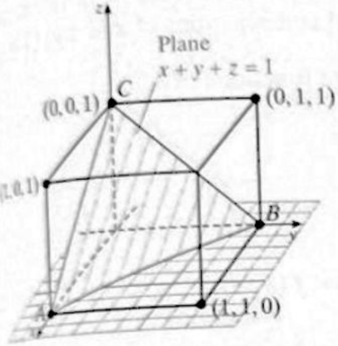


Figure 11.47 Find the hottest and coldest points on the plate inside the cube.

Solution

The cube and plate are shown in Figure 11.47. We will use Lagrange multipliers to find all critical points in the interior of the plate, and then we will examine the plate's boundary. To apply the method of Lagrange multipliers, we must solve $\nabla T = \lambda \nabla g$, where $g(x, y, z) = x + y + z$. We obtain the partial derivatives.

$$T_x = -4x \qquad T_y = -2y \qquad T_z = -2z \qquad g_x = g_y = g_z = 1$$

We must solve the system

$$\begin{cases} -4x = \lambda & \leftarrow T_x = \lambda g_x \\ -2y = \lambda & \leftarrow T_y = \lambda g_y \\ -2z = \lambda & \leftarrow T_z = \lambda g_z \\ x + y + z = 1 & \leftarrow g(x, y, z) = 1 \end{cases}$$

SMH See Problem 29 of Problem Set 3 of the *Student Mathematics Handbook*.

The solution of this system is $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$. The boundary of the plate is a triangle with vertices $A(1, 0, 0)$, $B(0, 1, 0)$, and $C(0, 0, 1)$. The temperature along the edges of this triangle may be found as follows:

Edge AC: $x + z = 1, y = 0$ $\qquad T_1(x) = 4 - 2x^2 - (0)^2 - (1 - x)^2 = 3 - 3x^2 + 2x, \quad 0 \le x \le 1$

Edge AB: $x + y = 1, z = 0$ $\qquad T_2(x) = 4 - 2x^2 - (1 - x)^2 - (0)^2 = 3 - 3x^2 + 2x, \quad 0 \le x \le 1$

Edge BC: $y + z = 1, x = 0$ $\qquad T_3(y) = 4 - 2(0)^2 - y^2 - (1 - y)^2 = 3 + 2y - 2y^2, \quad 0 \le y \le 1$

Edge AC: Differentiating, $T_1'(x) = T_2'(x) = -6x + 2$, which equals 0 when $x = \frac{1}{3}$. If $x = \frac{1}{3}$, then $z = \frac{2}{3}$ (because $x + z = 1$, $y = 0$ on edge AC), so we have the critical point $\left(\frac{1}{3}, 0, \frac{2}{3}\right)$.

Edge AB: Because $T_2 = T_1$, we see $x = \frac{1}{3}$. If $x = \frac{1}{3}$, then $y = \frac{2}{3}$ (because $x + y = 1$, $z = 0$ on edge BC), so we have another critical point $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$.

Edge BC: Differentiating, $T_3'(y) = 2 - 4y$, which equals 0 when $y = \frac{1}{2}$. Because $y + z = 1$ and $x = 0$, we have the critical point $\left(0, \frac{1}{2}, \frac{1}{2}\right)$.

Endpoints of the edges: $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

The last step is to evaluate T at the critical points and the endpoints:

$$T\left(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}\right) = 3\tfrac{3}{5};$$
$$T\left(\tfrac{1}{3}, 0, \tfrac{2}{3}\right) = 3\tfrac{1}{3}; \qquad T\left(\tfrac{1}{3}, \tfrac{2}{3}, 0\right) = 3\tfrac{1}{3}; \qquad T\left(0, \tfrac{1}{2}, \tfrac{1}{2}\right) = 3\tfrac{1}{2};$$
$$T(1, 0, 0) = 2; \qquad T(0, 1, 0) = 3; \qquad T(0, 0, 1) = 3$$

Comparing these values (remember that the temperature is in hundreds of degrees Celsius), we see that the highest temperature is 360°C at $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$ and the lowest temperature is 200°C at $(1, 0, 0)$. ■

Notice that the multiplier is used only as an intermediary device for finding the critical points and plays no role in the final determination of the constrained extrema. However, the value of λ is more important in certain problems, thanks to the interpretation given in the following theorem.

THEOREM 11.15 Rate of change of the extreme value

Suppose E is an extreme value (maximum or minimum) of f subject to the constraint $g(x, y) = c$. Then the Lagrange multiplier λ is the rate of change of E with respect to c; that is, $\lambda = dE/dc$.

Proof Note that at the extreme value (x, y) we have

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad \text{and} \quad g(x, y) = c$$

The coordinates of the optimal ordered pair (x, y) depend on c (because different constraint levels will generally lead to different optimal combinations of x and y). Thus,

$$E = E(x, y) \quad \text{where } x \text{ and } y \text{ are functions of } c$$

By the chain rule for partial derivatives:

$$\begin{aligned}
\frac{dE}{dc} &= \frac{\partial E}{\partial x}\frac{dx}{dc} + \frac{\partial E}{\partial y}\frac{dy}{dc} \\
&= f_x\frac{dx}{dc} + f_y\frac{dy}{dc} && \text{Because } E = f(x, y) \\
&= \lambda g_x\frac{dx}{dc} + \lambda g_y\frac{dy}{dc} && \text{Because } f_x = \lambda g_x \text{ and } f_y = \lambda g_y \\
&= \lambda\left(g_x\frac{dx}{dc} + g_y\frac{dy}{dc}\right) \\
&= \lambda\frac{dg}{dc} && \text{Chain rule} \\
&= \lambda && \text{Because } \frac{dg}{dc} = 1 \text{ (remember } g = c)
\end{aligned}$$

❑

This theorem can be interpreted as saying that the multiplier estimates the change in the extreme value E that results when the constraint c is increased by 1 unit. This interpretation is illustrated in the following example.

EXAMPLE 3 Maximum output for a Cobb-Douglas production function

If x thousand dollars is spent on labor, and y thousand dollars is spent on equipment, it is estimated that the output of a certain factory will be

$$Q(x, y) = 50x^{2/5}y^{3/5}$$

units. If \$150,000 is available, how should this capital be allocated between labor and equipment to generate the largest possible output? How does the maximum output change if the money available for labor and equipment is increased by \$1,000? In economics, an output function of the general form $Q(x, y) = x^{\alpha}y^{1-\alpha}$ is known as a *Cobb-Douglas production function*.

Solution

Because x and y are given in units of \$1,000, the constraint equation is $x + y = 150$. If we set $g(x, y) = x + y$, we wish to maximize Q subject to $g(x, y) = 150$. To apply the method of Lagrange multipliers, we first find

$$Q_x = 20x^{-3/5}y^{3/5} \qquad Q_y = 30x^{2/5}y^{-2/5} \qquad g_x = 1 \qquad g_y = 1$$

Next, solve the system

$$\begin{cases} 20x^{-3/5}y^{3/5} = \lambda(1) \\ 30x^{2/5}y^{-2/5} = \lambda(1) \\ x + y = 150 \end{cases}$$

From the first two equations we have

$$\begin{aligned} 20x^{-3/5}y^{3/5} &= 30x^{2/5}y^{-2/5} \\ 20y &= 30x \\ y &= 1.5x \end{aligned}$$

Substitute $y = 1.5x$ into the equation $x + y = 150$ to find $x = 60$. This leads to the solution $y = 90$, so that the maximum output is

$$Q(60, 90) = 50(60)^{2/5}(90)^{3/5} \approx 3,826.273502 \quad \text{units}$$

We also find that

$$\lambda = 20(60)^{-3/5}(90)^{3/5} \approx 25.50849001$$

Thus, the maximum output is about 3,826 units and occurs when \$60,000 is allocated to labor and \$90,000 to equipment. We also note that an increase of \$1,000 (1 unit) in the available funds will increase the maximum output by approximately $\lambda \approx 25.51$ units (from 3,826.27 to 3,851.78 units). ■

LAGRANGE MULTIPLIERS WITH TWO PARAMETERS

The method of Lagrange multipliers can also be applied in situations with more than one constraint equation. Suppose we wish to locate an extremum of a function defined by $f(x, y, z)$ subject to two constraints, $g(x, y, z) = c_1$ and $h(x, y, z) = c_2$, where g and h are also differentiable and ∇g and ∇h are not parallel. By generalizing Lagrange's theorem, it can be shown that if (x_0, y_0, z_0) is the desired extremum, then there are numbers λ and μ such that $g(x_0, y_0, z_0) = c_1$, $h(x_0, y_0, z_0) = c_2$, and

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

As in the case of one constraint, we proceed by first solving this system of equations simultaneously to find λ, μ, x_0, y_0, z_0 and then evaluating $f(x, y, z)$ at each solution and comparing to find the required extremum. This approach is illustrated in our final example of this section.

EXAMPLE 4 Optimization with two constraints

Find the point on the intersection of the plane $x + 2y + z = 10$ and the paraboloid $z = x^2 + y^2$ that is closest to the origin (see Figure 11.48). You may assume that such a point exists.

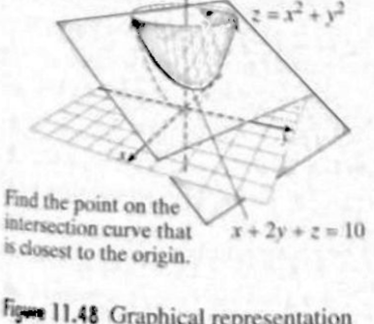


Figure 11.48 Graphical representation of Example 4

Solution

The distance from a point (x, y, z) to the origin is $s = \sqrt{x^2 + y^2 + z^2}$, but instead of minimizing this quantity, it is easier to minimize its square. That is, we will minimize

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the joint constraints

$$g(x, y, z) = x + 2y + z = 10 \quad \text{and} \quad h(x, y, z) = x^2 + y^2 - z = 0$$

Compute the partial derivatives of f, g, and h:

$$\begin{array}{lll} f_x = 2x & f_y = 2y & f_z = 2z \\ g_x = 1 & g_y = 2 & g_z = 1 \\ h_x = 2x & h_y = 2y & h_z = -1 \end{array}$$

To apply the method of Lagrange multipliers, we use the formula

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

which leads to the following system of equations:

$$\begin{cases} 2x = \lambda(1) + \mu(2x) \\ 2y = \lambda(2) + \mu(2y) \\ 2z = \lambda(1) + \mu(-1) \\ x + 2y + z = 10 \\ z = x^2 + y^2 \end{cases}$$

This is not a linear system, so solving it requires ingenuity.

Multiply the first equation by 2 and subtract the second equation to obtain

$$4x - 2y = (4x - 2y)\mu$$
$$(4x - 2y) - (4x - 2y)\mu = 0$$
$$(4x - 2y)(1 - \mu) = 0$$
$$4x - 2y = 0 \quad \text{or} \quad 1 - \mu = 0$$

CASE I: **If $4x - 2y = 0$**, then $y = 2x$. Substitute this into the two constraint equations:

$x + 2y + z = 10$	$x^2 + y^2 - z = 0$
$x + 2(2x) + z = 10$	$x^2 + (2x)^2 - z = 0$
$z = 10 - 5x$	$z = 5x^2$

By substitution we have $5x^2 = 10 - 5x$, which has solutions $x = 1$ and $x = -2$. This implies

$x = 1$	$x = -2$
$y = 2x = 2(1) = 2$	$y = 2x = 2(-2) = -4$
$z = 5x^2 = 5(1)^2 = 5$	$z = 5x^2 = 5(-2)^2 = 20$

Thus, the points $(1, 2, 5)$ and $(-2, -4, 20)$ are candidates for the minimal distance.

CASE II: **If $1 - \mu = 0$**, then $\mu = 1$, and we look at the system of equations involving x, y, z, λ, and μ.

$$\begin{cases} 2x = \lambda(1) + \mu(2x) \\ 2y = \lambda(2) + \mu(2y) \\ 2z = \lambda(1) + \mu(-1) \\ x + 2y + z = 10 \\ z = x^2 + y^2 \end{cases}$$

The top equation becomes $2x = \lambda + 2x$, so that $\lambda = 0$. We now find z from the third equation:

$$2z = -1 \quad \text{or} \quad z = -\tfrac{1}{2}$$

Next, turn to the constraint equations:

$x + 2y + z = 10$	$x^2 + y^2 - z = 0$
$x + 2y - \frac{1}{2} = 10$	$x^2 + y^2 + \frac{1}{2} = 0$
$x + 2y = 10 + \frac{1}{2}$	$x^2 + y^2 = -\frac{1}{2}$

There is no solution because $x^2 + y^2$ cannot equal a negative number.

We check the candidates for the minimal distance:

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \text{so that}$$
$$f(1, 2, 5) = 1^2 + 2^2 + 5^2 = 30$$
$$f(-2, -4, 20) = (-2)^2 + (-4)^2 + 20^2 = 420$$

Because $f(x, y, z)$ represents the square of the distance, the minimal distance is $\sqrt{30}$ and the point on the intersection of the two surfaces nearest to the origin is $(1, 2, 5)$. ■

A GEOMETRIC INTERPRETATION

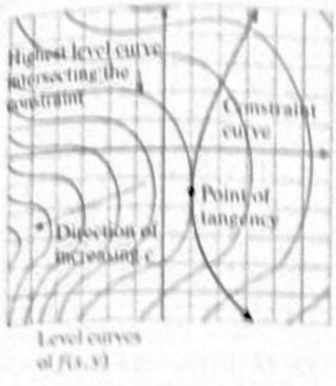


Figure 11.49 Increasing level curves and the constraint curve

Lagrange's theorem can be interpreted geometrically. Suppose the constraint curve $g(x, y) = c$ and the level curves $f(x, y) = k$ are drawn in the xy-plane, as shown in Figure 11.49.

To maximize $f(x, y)$ subject to the constraint $g(x, y) = c$, we must find the "highest" (leftmost, actually) level curve of f that intersects the constraint curve. As Figure 11.49 suggests, this critical intersection occurs at a point where the constraint curve is tangent to a level curve—that is, where the slope of the constraint curve $g(x, y) = c$ is equal to the slope of a level curve $f(x, y) = k$. According to the formula derived in Section 11.5 (p. 732),

$$\text{Slope of constraint curve } g(x, y) = c \text{ is } \frac{-g_x}{g_y}.$$

$$\text{Slope of each level curve is } \frac{-f_x}{f_y}.$$

The condition that the slopes are equal can be expressed by

$$\frac{-f_x}{f_y} = \frac{-g_x}{g_y}, \qquad \text{or, equivalently,} \qquad \frac{f_x}{g_x} = \frac{f_y}{g_y}$$

Let λ equal this common ratio,

$$\lambda = \frac{f_x}{g_x} \qquad \text{and} \qquad \lambda = \frac{f_y}{g_y}$$

so that

$$f_x = \lambda g_x \qquad \text{and} \qquad f_y = \lambda g_y$$

and

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} = \lambda(g_x\mathbf{i} + g_y\mathbf{j}) = \lambda \nabla g$$

Because the point in question must lie on the constraint curve, we also have $g(x, y) = c$. If these equations are satisfied at a certain point (a, b), then f will reach its constrained *maximum* at (a, b) if the *highest* level curve that intersects the constraint curve does so at this point. On the other hand, if the *lowest* level curve that intersects the constraint curve does so at (a, b), then f achieves its constrained *minimum* at this point.

11.8 PROBLEM SET

In the problems in this set, you may assume that the requested extreme value(s) exist.

Use the method of Lagrange multipliers to find the required constrained extrema in Problems 1–14.

1. Maximize $f(x, y) = xy$ subject to $2x + 2y = 5$.
2. Maximize $f(x, y) = xy$ subject to $x + y = 20$.
3. Maximize $f(x, y) = 16 - x^2 - y^2$ subject to $x + 2y = 6$.
4. Minimize $f(x, y) = x^2 + y^2$ subject to $x + y = 24$.
5. Minimize $f(x, y) = x^2 + y^2$ subject to $xy = 1$.
6. Minimize $f(x, y) = x^2 - xy + 2y^2$ subject to $2x + y = 22$.
7. Minimize $f(x, y) = x^2 - y^2$ subject to $x^2 + y^2 = 4$.
8. Maximize $f(x, y) = x^2 - 2y - y^2$ subject to $x^2 + y^2 = 1$.
9. Maximize $f(x, y) = \cos x + \cos y$ subject to $y = x + \frac{\pi}{4}$.
10. Maximize $f(x, y) = e^{xy}$ subject to $x^2 + y^2 = 3$.
11. Maximize $f(x, y) = \ln(xy^2)$ subject to $2x^2 + 3y^2 = 8$ for $x > 0$.
12. Maximize $f(x, y, z) = xyz$ subject to $3x + 2y + z = 6$.
13. Minimize $f(x, y, z) = x^2+y^2+z^2$ subject to $x-2y+3z = 4$.
14. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to $4x^2 + 2y^2 + z^2 = 4$.
15. Find the smallest value of $f(x, y, z) = 2x^2+4y^2+z^2$ subject to $4x - 8y + 2z = 10$. What, if anything, can be said about the largest value of f subject to this constraint?
16. Let $f(x, y, z) = x^2y^2z^2$. Show that the maximum value of f on the sphere $x^2 + y^2 + z^2 = R^2$ is $R^6/27$.
17. Find the maximum and minimum values of $f(x, y, z) = x - y + z$ on the sphere $x^2 + y^2 + z^2 = 100$.
18. Find the maximum and minimum values of $f(x, y, z) = 4x - 2y - 3z$ on the sphere $x^2 + y^2 + z^2 = 100$.
19. Use Lagrange multipliers to find the distance from the origin to the plane $Ax + By + Cz = D$ where at least one of A, B, C is nonzero.