Ex) Solve: $\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$

Sol^n:-

Given $\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$

$= e^y\left(e^x + x^2 e^{x^3}\right)$

$\Rightarrow e^{-y}\,dy = \left(e^x + x^2 e^{x^3}\right)dx \quad \text{---}(1)$

Integrating (1),

$\int e^{-y}\,dy = \int \left(e^x + x^2 e^{x^3}\right)dx + c$

$\Rightarrow -e^{-y} = e^x + \int x^2 e^{x^3}\,dx + c$

$\Rightarrow -e^{-y} = e^x + \frac{1}{3}\int e^t\,dt + c$

Putting $x^3 = t$
$\Rightarrow 3x^2\,dx = dt$

$\Rightarrow -e^{-y} = e^x + \frac{1}{3}e^t + c$

$\Rightarrow -e^{-y} = e^x + \frac{1}{3}e^{x^3} + c$ //

Note: Equations of the form

$\frac{dy}{dx} = f(ax+by+c)$ or $\frac{dy}{dx} = f(ax+by)$

can be reduced to an equation in which variables can be separated. For this purpose we use the substitution $ax+by+c = v$ or $ax+by = v$.

Ex) Solve $\frac{dy}{dx} = (4x+y+1)^2$

Sol^n:

let $4x+y+1=v$ ....(1)

Diff. (1) w.r.t x, we get

$$4+\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 4 \quad ....(2)$$

$\therefore$ (1)+(2) $\Rightarrow$ given eq^n becomes

$$\frac{dv}{dx} - 4 = v^2$$

$$\Rightarrow \frac{dv}{dx} = v^2+4$$

$$\Rightarrow \frac{dv}{v^2+4} = dx$$

Integrating, we get,

$$\int \frac{dv}{v^2+4} = \int dx + c$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\frac{v}{2} = x+c$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x+c$$

$$\Rightarrow \tan^{-1}\left(\frac{4x+y+1}{2}\right) = 2x+2c$$

$$\Rightarrow \frac{4x+y+1}{2} = \tan(2x+2c)$$

$$\Rightarrow 4x+y+1 = 2\tan(2x+2c) \ //$$

Ex) Solve $(x+y)^2 \frac{dy}{dx} = a^2$

Sol^n: let $x+y=v$ ...(1)

Diff. (1) w.r.t. x, we get,

$$1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1 \quad ..(2)$$

$\therefore$ given eqn becomes

$$v^2\left(\frac{dv}{dx}-1\right) = a^2$$

$$\Rightarrow v^2 \frac{dv}{dx} = a^2 + v^2$$

$$\Rightarrow dx = \frac{v^2}{a^2+v^2}\, dv$$

$$\Rightarrow \int dx = \int \frac{v^2}{a^2+v^2}\, dv + c$$

$$\Rightarrow x = \int \left(1 - \frac{a^2}{a^2+v^2}\right) dv + c$$

$$\Rightarrow x = v - a^2 \int \frac{dv}{a^2+v^2} + c$$

$$\Rightarrow x = v - a^2 \cdot \frac{1}{a} \tan^{-1}\frac{v}{a} + c$$

$$\Rightarrow x = x + y - a\tan^{-1}\left(\frac{x+y}{a}\right) + c$$

$$\Rightarrow y + c = a\tan^{-1}\left(\frac{x+y}{a}\right) \quad //$$

Ex) Solve: $(x+y)(dx - dy) = dx + dy$

Soln: Given eqn is

$$(x+y-1)\,dx = (x+y+1)\,dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y-1}{x+y+1} \quad \text{---(1)}$$

Let $x+y = v$

$$\Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$\therefore$ (1) $\Rightarrow$

$$\frac{dv}{dx} - 1 = \frac{v-1}{v+1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{v-1}{v+1} + 1 = \frac{2v}{v+1}$$

$$\Rightarrow 2\,dx = \left(\frac{v+1}{v}\right) dv$$

$$\Rightarrow 2\,dx = \left(1 + \frac{1}{v}\right) dv$$

$$\Rightarrow 2\int dx = \int \left(1 + \frac{1}{v}\right) dv + c$$

$\Rightarrow 2x = v + \log v + c$

$\Rightarrow 2x = x + y + \log(x+y) + c$

$\Rightarrow x = y + \log(x+y) + c$ //