Homogeneous Equations:

A differential equation of first degree and first order is said to be homogeneous if it can be put in the form.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \cdots\cdots (1)$$

To solve (1), let $\frac{y}{x} = v$ i.e. $y = xv$ $\quad \cdots (2)$

Differentiating w.r.t x, (2) gives

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{———} \cdots (3)$$

using (2) and (3), (1) becomes

$$v + x\frac{dv}{dx} = f(v)$$

$$\Rightarrow x\frac{dv}{dx} = f(v) - v$$

Separating the variables x and v, we have

$$\frac{dx}{x} = \frac{dv}{f(v) - v}$$

$\therefore \quad \log x + c = \int \frac{dv}{f(v) - v}$, where c is an arbitrary constant. After integration, replace v by $\frac{y}{x}$.

Ex) Solve:

$$x^2\,dy + y(x+y)\,dx = 0$$

Sol^n: we have,

$$x^2\,dy + y(x+y)\,dx = 0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{xy + y^2}{x^2}\right)$$

$$= -\frac{y}{x} - \frac{y^2}{x^2} \quad \cdots (1)$$

Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ (1) $\Rightarrow$

$$v + x\frac{dv}{dx} = -v - v^2$$

$$\Rightarrow x\frac{dv}{dx} = -2v - v^2$$

$$\Rightarrow \frac{dv}{v^2+2v} = -\frac{dx}{x}$$

$$\Rightarrow \left\{\frac{1}{v(v+2)}\right\} dv = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{v} - \frac{1}{v+2}\right) dv = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\int \frac{1}{v}\,dv - \frac{1}{2}\int \frac{1}{v+2}\,dv = -\int \frac{dx}{x} + c$$

$$\Rightarrow \frac{1}{2}\log v - \frac{1}{2}\log(v+2) = -\log x + c$$

$$\Rightarrow \log v - \log(v+2) = -2\log x + 2c$$

$$\Rightarrow \log\frac{v}{v+2} + \log x^2 = 2c$$

$$\Rightarrow \log\frac{vx^2}{v+2} = \log a \qquad \text{where } 2c = \log a$$

$$\Rightarrow \frac{vx^2}{v+2} = a \quad \Rightarrow \frac{\frac{y}{x}\cdot x^2}{\frac{y}{x}+2} = a$$

$$\Rightarrow \frac{yx^2}{y+2x} = a$$

$$\Rightarrow yx^2 = a(y+2x)$$

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Ex) Solve: $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$

Soln: We have

$$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x} \quad \text{--- (1)}$$

Put $y = vx$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore (1) \Rightarrow$

$$v + x\frac{dv}{dx} = v + \tan v$$

$$\Rightarrow x\frac{dv}{dx} = \tan v$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{\tan v}$$

$$\Rightarrow \frac{dx}{x} = \frac{\cos v}{\sin v}\, dv$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{\cos v}{\sin v}\, dv + \log c$$

$$\Rightarrow \log x = \log \sin v + \log c$$

$$\Rightarrow x = c \sin v$$

$$\Rightarrow x = c \sin\left(\frac{y}{x}\right)$$

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