Separation of Variables:

If a differential equation of the first order and first degree is of the form

$$f_1(x)\,dx = f_2(y)\,dy \quad \text{---- (1)}$$

where $f_1(x)$ is a function of x only and $f_2(y)$ is a function of y only, then we say that the variables are separable in the given differential equation. Such equations are solved by integrating both sides of (1) and adding an arbitrary constant of integration to any one of the two sides. Thus the solution of (1) is

$$\int f_1(x)\,dx = \int f_2(y)\,dy + c, \quad \text{---- (2)}$$

Note: To simplify the solution (2), the constant of integration c can be selected in any suitable form. For example, c can be replaced by $\frac{c}{4}$, $\log c$, $\sin c$, $\tan c$, e^c, $\tan^{-1} c$ etc.

Solve:

(i) $(x+1)\frac{dy}{dx} = x(y^2+1)$

(ii) $(xy^2+x)\,dy + (yx^2+y)\,dy = 0$

(iii) $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

(i) Sol^n: we have

$$(x+1)\frac{dy}{dx} = x(y^2+1)$$

$$\Rightarrow (x+1)\,dy = x(y^2+1)\,dx$$

$$\Rightarrow \frac{dy}{y^2+1} = \frac{x\,dx}{x+1}$$

$$\Rightarrow \frac{dy}{y^2+1} = \frac{(x+1-1)}{(x+1)}\,dx$$

$$\Rightarrow \frac{dy}{y^2+1} = \left(1 - \frac{1}{x+1}\right)dx$$

on integrating we get,

$$\int \frac{dy}{y^2+1} = \int dx - \int \frac{dx}{x+1} + c$$

$$\Rightarrow \tan^{-1} y = x - \log(x+1) + c \quad //$$

(ii) Sol^n: we have,

$$(xy^2+x)\,dx + (yx^2+y)\,dy = 0$$

$$\Rightarrow (y^2+1)x\,dx = -(x^2+1)y\,dy$$

$$\Rightarrow \frac{x}{(x^2+1)}\,dx = -\frac{y}{(y^2+1)}\,dy.$$

$$\Rightarrow \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = -\frac{1}{2}\int \frac{2y}{y^2+1}\,dy + \frac{1}{2}c$$

$$\Rightarrow \frac{1}{2}\log(x^2+1) = -\frac{1}{2}\log(y^2+1) + \frac{1}{2}c$$

$$\Rightarrow \log(x^2+1) = -\log(y^2+1) + \log c'$$

$$\Rightarrow \log(x^2+1) + \log(y^2+1) = \log c'$$

$$\Rightarrow (x^2+1)(y^2+1) = c'$$

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(iii) Soln:

we have,

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$$

$$\Rightarrow (\sin y + y\cos y)\, dy = x(2\log x + 1)\, dx$$

$$\Rightarrow \int (\sin y + y\cos y)\, dy = \int \{x(2\log x + 1)\}\, dx + c$$

$$\Rightarrow -\cos y + y\sin y - \int 1.\sin y\, dy = 2\int \log x . x\, dx + \int x\, dx + c$$

$$\Rightarrow -\cos y + y\sin y + \cos y = 2\log x . \frac{x^2}{2} - 2\int \frac{1}{x} \cdot \frac{x^2}{2}\, dx + \frac{x^2}{2} + c$$

$$\Rightarrow y\sin y = 2\log x . \frac{x^2}{2} - \int x\, dx + \frac{x^2}{2} + c$$

$$\Rightarrow y\sin y = 2\log x \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^2}{2} + c$$

$$\Rightarrow y\sin y = x^2 \log x + c$$

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