Remark:
$$T = \{r \in \mathbb{Q} : 0 \le r, r^2 < 2\}$$
Then we can conclude that $y = \sup T$ satisfies $y^2 = 2$. i.e. $y = \sqrt{2}$. But we know $\sqrt{2} \notin \mathbb{Q}$. Thus the set T consists of rational numbers doesn't have a supremum belonging to the set $\mathbb{Q}$. The ~~the~~ ordered field $\mathbb{Q}$ of rational numbers does not possess the Complete-
-ness property.

The Density Theorem: If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

Pf:- It is no loss of generality to assume that $x > 0$. Since $x < y$ i.e. $y - x > 0$, so there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$.

$$\therefore 1 < ny - nx \Rightarrow nx + 1 < ny \quad \text{— (I)}$$

Also, $nx > 0$, so by Corollary of Archimedian property, we get $m \in \mathbb{N}$ such that

$$m - 1 \le nx < m \quad \text{— (II)}$$

$$\therefore m \le nx + 1 < m + 1 \quad \text{— (III)}$$

from (i) & (iii)

$$m \le nx + 1 < ny$$

Thus we get,

$$nx < m < ny$$

$$\Rightarrow x < \frac{m}{n} < y$$

$$\Rightarrow x < r < y, \quad r = \frac{m}{n}.$$

Corollary. If x and y are real numbers with $x < y$, Then there exists an irrational numbers z such that $x < z < y$.

Pf. Let x and y be any real numbers with $x < y$.

Then $x/\sqrt{2}$ and $y/\sqrt{2}$ are also real numbers

and $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$

So by Density Theorem, there exists a rational number r such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$

$$\Rightarrow x < r\sqrt{2} < y$$

$$\Rightarrow x < z < y, \quad z = r\sqrt{2}$$

As r is rational and $\sqrt{2}$ is irrational so $z = r\sqrt{2}$ is irrational #