Corollary: If $t > 0$, then exists $n_t \in \mathbb{N}$ such that $0 < \frac{1}{n_t} < t$.

Pf. Since $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$ and $t > 0$, then t is not a lower bound for the set $\{\frac{1}{n} : n \in \mathbb{N}\}$. So, there exists $n_t \in \mathbb{N}$ such that $0 < \frac{1}{n_t} < t$.

Corollary: If $y > 0$, then exists $n_y \in \mathbb{N}$ such that $n_y - 1 \leq y < n_y$.

Pf: Consider a set $E_y = \{m \in \mathbb{N} : y < m\}$

By Archimedian property for $y \in \mathbb{R}$, then exists a natural number m such that $y < m$. So the set E_y is nonempty and it is a subset of $\mathbb{N}$, the set of natural numbers.

By well-ordering principle E_y has a least element say n_y. Then $y < n_y$ [$\because n_y \in E_y$]

Also, since $n_y - 1 < n_y$ so $n_y - 1 \notin E_y$ as n_y is the least element of E_y.

$$\therefore n_y - 1 \leq y < n_y.$$

of at least ...

2.4.7 Theorem *There exists a positive real number x such that $x^2 = 2$.*

Proof. Let $S := \{s \in \mathbb{R}: 0 \le s, s^2 < 2\}$. Since $1 \in S$, the set is not empty. Also, S is bounded above by 2, because if $t > 2$, then $t^2 > 4$ so that $t \notin S$. Therefore the Supremum Property implies that the set S has a supremum in $\mathbb{R}$, and we let $x := \sup S$. Note that $x > 1$.

We will prove that $x^2 = 2$ by ruling out the other two possibilities: $x^2 < 2$ and $x^2 > 2$.

First assume that $x^2 < 2$. We will show that this assumption contradicts the fact that $x = \sup S$ by finding an $n \in \mathbb{N}$ such that $x + 1/n \in S$, thus implying that x is not an upper bound for S. To see how to choose n, note that $1/n^2 \le 1/n$ so that

$$\left(x + \frac{1}{n}\right)^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2} \le x^2 + \frac{1}{n}(2x + 1).$$

Hence if we can choose n so that

$$\frac{1}{n}(2x + 1) < 2 - x^2,$$

then we get $(x + 1/n)^2 < x^2 + (2 - x^2) = 2$. By assumption we have $2 - x^2 > 0$, so that $(2 - x^2)/(2x + 1) > 0$. Hence the Archimedean Property (Corollary 2.4.5) can be used to obtain $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \frac{2 - x^2}{2x + 1}.$$

These steps can be reversed to show that for this choice of n we have $x + 1/n \in S$, which contradicts the fact that x is an upper bound of S. Therefore we cannot have $x^2 < 2$.

Now assume that $x^2 > 2$. We will show that it is then possible to find $m \in \mathbb{N}$ such that $x - 1/m$ is also an upper bound of S, contradicting the fact that $x = \sup S$. To do this, note that

$$\left(x - \frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2} > x^2 - \frac{2x}{m}.$$

Hence if we can choose m so that

$$\frac{2x}{m} < x^2 - 2,$$

then $(x - 1/m)^2 > x^2 - (x^2 - 2) = 2$. Now by assumption we have $x^2 - 2 > 0$, so that $(x^2 - 2)/2x > 0$. Hence, by the Archimedean Property, there exists $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \frac{x^2 - 2}{2x}.$$

These steps can be reversed to show that for this choice of m we have $(x - 1/m)^2 > 2$. Now if $s \in S$, then $s^2 < 2 < (x - 1/m)^2$, whence it follows from 2.1.13(a) that $s < x - 1/m$. This implies that $x - 1/m$ is an upper bound for S, which contradicts the fact that $x = \sup S$. Therefore we cannot have $x^2 > 2$.

Since the possibilities $x^2 < 2$ and $x^2 > 2$ have been excluded, we must have $x^2 = 2$.

Q.E.D.