

Ex) Given $\log 100 = 2$, $\log 101 = 2.0043$, $\log 103 = 2.0128$,
 $\log 104 = 2.0170$, find $\log 102$.

Sol:- Let $y = f(x) = \log x$

Here the data can be put as follows with $\Delta x = h$

$x:$	100	101	102	103	104
$f(x) = \log x:$	2	2.0043	?	2.0128	2.0170

Since four values of $f(x)$ are given at $\Delta x = h$

$$\therefore \Delta^4 f(x) = 0 \text{ at } x$$

Putting $x = 100$, we get, $\Delta^4 f(100) = 0$

$$\Delta^4 f(100) = 0$$

$$\Rightarrow (-1)^4 f(100) = 0$$

$$\Rightarrow [f(104) - 4f(103) + 6f(102) - 4f(101) + f(100)] = 0$$

$$\Rightarrow 6 \log 102 = 12.0514$$

$$\Rightarrow \log 102 = 2.0086 \text{ (approx)}$$

* Newton-Gregory forward difference interpolation formula:

If $a, a+h, a+2h, \dots, a+mh$ are $\frac{1}{h}$ equi-spaced

values of argument x so that a function $y = f(x)$ assumes the values $f(a), f(a+h), f(a+2h), \dots, f(a+mh)$, then

$$f(a+mh) = f(a) + \frac{\Delta f(a)}{1!} + \frac{\Delta^2 f(a)}{2!} + \dots + \frac{\Delta^m f(a)}{m!}$$

$$\text{where } \frac{\Delta^n f(a)}{n!} = \frac{f(a+nh) - f(a+(n-1)h)}{(n-1)!} = \frac{f(a+nh) - f(a+(n-1)h)}{h} + \dots + \frac{f(a+2h) - f(a+h)}{h} + f(a+h) - f(a)$$

Proof: We assume that the given $f(x)$ can be expressed as a polynomial of degree m in x .

Representing this in factorial notations,

$$\begin{aligned} f(x) &= A_0 + A_1(n-a)^{(1)} + A_2(n-a)^{(2)} + \dots + A_m(n-a)^{(m)} \\ &= A_0 + A_1(n-a) + A_2(n-a)\{n-(a+h)\} + \dots \\ &\quad \dots + A_m(n-a)\{n-(a+h)\} \dots [n-\{a+(m-1)h\}] \end{aligned}$$

where $A_0, A_1, A_2, \dots, A_m$ are constants.

To find these constants, put $n=a, a+h, a+2h, \dots, a+mh$ in (1).

\therefore for $n=a$, the first terms of LHS are $\Delta^m f(a)$

$$(1) \Rightarrow f(a) = A_0 + aA_1 + (a+h)A_2 + \dots + (a+mh)A_m$$

for $n=a+h$,

$$(1) \Rightarrow f(a+h) = A_0 + hA_1 + \frac{f(a+h) - f(a)}{h} = \frac{\Delta f(a)}{h} + A_1$$

for $n=a+2h$,

$$\begin{aligned} (1) \Rightarrow f(a+2h) &= A_0 + 2hA_1 + 2h \cdot h \cdot A_2 \\ &\Rightarrow A_2 = \frac{[f(a+2h) - 2\{f(a+h) - f(a)\} - f(a)]}{2h} \\ &= \frac{\Delta^2 f(a)}{2h} = \frac{\Delta^2 f(a)}{12 \cdot h^2} \end{aligned}$$

and so on, all other coefficients are zero.

$$\therefore \text{Thus } A_m = \frac{1}{m! h^m} \Delta^m f(a)$$

Substituting these values of $A_0, A_1, A_2, \dots, A_m$ in (1), we get

$$\begin{aligned} f(x) &= f(a) + \frac{\Delta f(a)}{h} (n-a) + \frac{\Delta^2 f(a)}{2h} (n-a)\{n-(a+h)\} + \dots \\ &\quad + \frac{\Delta^m f(a)}{m! h^m} (n-a)\{n-(a+h)\} \dots [n-\{a+(m-1)h\}] \end{aligned}$$

Putting $\frac{n-a}{h} = m$ (say)

$$\Rightarrow x = a + mh$$

$$f(a+mh) = f(a) + \frac{m^{(1)}}{1!} \Delta f(a) + \frac{m^{(2)}}{2!} \Delta^2 f(a) + \dots + \frac{m^{(m)}}{m!} \Delta^m f(a)$$

This is Newton-Gregory formula for forward interpolation.

This formula is used mainly for interpolating the values of y near the beginning ($x > a$) of a set of tabulated values.

Ex. From the following table, find the number of students who obtained less than 45 marks: (1973) June (Q. 2)

Marks:	30-40	40-50	50-60	60-70	70-80
No. of students;	31	42	51	35	31

Soln:- The table of finite differences is given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
Below 40	31				
" 50	42	$73 + \Delta 42$	$4(11) + 9$	$4(11) - (42) + 9$	-25
" 60	51	$124 - 73$	$6(18) - 16(12) + 22$	$6(18) - 2(12) + 37$	
" 70	35	$159 - 124$	$3(12) - 21$	$3(12) - 37$	
" 80	31	$190 - 159$	-4		

Now, from Newton-Gregory formula for forward interpolation, we have,

$$f(a+mh) = f(a) + m \Delta f(a) + \frac{m(m-1)}{12} \Delta^2 f(a) + \frac{m(m-1)(m-2)}{13} \Delta^3 f(a) \\ + \frac{m(m-1)(m-2)(m-3)}{14} \Delta^4 f(a) \quad \dots \dots (1)$$

Here $h = 10$, $a = 40$; we have to interpolate $f(45)$

$$\therefore a+mh = 45 \quad \Rightarrow \quad m = \frac{45-40}{10} = \frac{1}{2}$$

$$f(45) = f(40) + \frac{1}{2} \Delta f(40) + \frac{1}{2} \frac{(-\frac{1}{2})}{12} \Delta^2 f(40) \\ + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{13} \Delta^3 f(40) + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{14} \Delta^4 f(40) \\ = 31 + \frac{1}{2} (42) + \frac{1}{8} (9) + \frac{1}{16} (-25) - \frac{5}{128} (37) \\ = 31 + 21 - 1.125 + 1.5625 - 1.4453 \\ = 47.8672 \approx 48 (\text{approx})$$

Therefore the number of students who obtained less than 45 marks is 48.