3) If the given equation $M\,dx + N\,dy = 0$ is homogeneous such that $Mx + Ny \neq 0$, then $\dfrac{1}{Mx+Ny}$ is an integrating factor.

Example:
$$\frac{dy}{dx} = \frac{x^2+y^2}{xy}$$

$$\therefore \quad (x^2+y^2)\,dx - xy\,dy = 0 \quad \text{---}(1)$$

Here $M = x^2+y^2$ and $N = -xy$

Now,
$$Mx + Ny = x^3 + xy^2 - xy^2 = x^3 \neq 0$$

Hence
$$I.F = \frac{1}{x^3}$$

Multiplying ~~the~~ eqn (1) both sides by $\frac{1}{x^3}$, we get
$$\left(\frac{1}{x} + \frac{y^2}{x^3}\right) dx - \frac{y}{x^2}\,dy = 0$$

which is an exact differential equation.

4) If the differential equation $M\,dx + N\,dy = 0$ is not exact, then it can be made exact by multiplying $x^\alpha y^\beta$ both sides. α and β are found out from $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Example:

$$(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$$

Here $M = y^3 - 2x^2y$ and $N = 2xy^2 - x^3$

$$\frac{\partial M}{\partial y} = 3y^2 - 2x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 2y^2 - 3x^2$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Multiplying both sides the above differential eq^n by $x^\alpha y^\beta$, so that it reduces in the form of

$M_1\,dx + N_1\,dy = 0$

where $M_1 = x^\alpha y^{\beta+3} - 2x^{\alpha+2}y^{\beta+1}$

and $N_1 = 2x^{\alpha+1}y^{\beta+2} - x^{\alpha+3}y^\beta$

Now, $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

$$\Rightarrow x^\alpha.(\beta+3)y^{\beta+2} - 2x^{\alpha+2}(\beta+1)y^\beta$$

$$= 2(\alpha+1)x^\alpha y^{\beta+2} - (\alpha+3)x^{\alpha+2}y^\beta$$

Dividing both sides by $x^\alpha y^\beta$ we get

$$(\beta+3)y^2 - 2x^2(\beta+1) = 2(\alpha+1)y^2 - (\alpha+3)x^2$$

$$\Rightarrow (\alpha+3-2\beta-2)x^2 + (\beta+3-2\alpha-2)y^2 = 0$$

$$\Rightarrow (\alpha-2\beta+1)x^2 + (\beta-2\alpha+1)y^2 = 0$$

$$\therefore \quad \alpha - 2\beta + 1 = 0 \quad \text{and} \quad \beta - 2\alpha + 1 = 0$$

$$\Rightarrow \alpha = 1, \ \beta = 1$$

Hence $IF = xy$