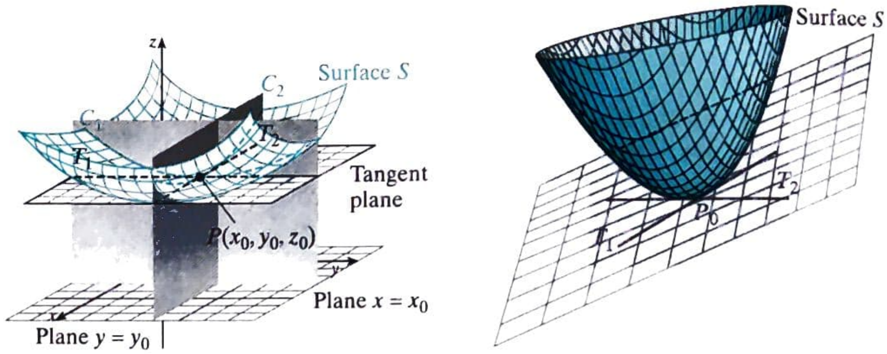
11.4 Tangent Planes, Approximations, and Differentiability

IN THIS SECTION tangent planes, incremental approximations, the total differential, differentiability

TANGENT PLANES

Suppose S is a surface with the equation $z = f(x, y)$, where f has continuous first partial derivatives f_x and f_y. Let $P_0(x_0, y_0, z_0)$ be a point on S, and let C_1 be the curve of intersection of S with the plane $x = x_0$ and C_2, the intersection of S with the plane $y = y_0$, as shown in Figure 11.20a. The tangent lines T_1 and T_2 to C_1 and C_2, respectively, at P_0 determine a unique plane, and in Section 11.6 we will find that this plane actually contains the tangent to every smooth curve on S that passes through P_0. We call this plane the **tangent plane** to S at P_0 (Figure 11.20b).



a. C_1 and C_2 are the curves of intersection of the surface S with the planes $x = x_0$ and $y = y_0$.

b. The tangent plane to S at P_0 contains the tangent lines T_1 and T_2 to C_1 and C_2, respectively.

Figure 11.20 Tangent plane to the surface S at the point P_0

To find an equation for the tangent plane, recall that the equation of a plane with normal $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

If $C \neq 0$, divide both sides by C and let $a = -A/C$ and $b = -B/C$ to obtain

$$z - z_0 = a(x - x_0) + b(y - y_0)$$

The intersection of this plane and the plane $x = x_0$ is the tangent line T_1, which we know has slope $f_y(x_0, y_0)$ from the geometric interpretation of partial derivatives. Setting $x = x_0$ in the equation for the tangent plane, we find that T_1 has the point-slope form

$$z - z_0 = b(y - y_0)$$

so we must have $b = f_y(x_0, y_0) = \left.\dfrac{\partial z}{\partial y}\right|_{(x_0, y_0)}$. Similarly, setting $y = y_0$, we obtain

$$z - z_0 = a(x - x_0)$$

which represents the tangent line T_2, with slope $a = f_x(x_0, y_0)$. To summarize:

Equation of the Tangent Plane

Suppose S is a surface with the equation $z = f(x, y)$ and let $P_0(x_0, y_0, z_0)$ be a point on S at which a tangent plane exists. Then an **equation for the tangent plane** to S at P_0 is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

If the equation is written in the form

$$Ax + By + Cz + D = 0$$

we say the equation of the plane is in standard form.

EXAMPLE 1 Equation of a tangent plane for a surface defined by $z = f(x, y)$

Find an equation for the tangent plane to the surface $z = \tan^{-1}\dfrac{y}{x}$ at the point $P_0\left(1, \sqrt{3}, \frac{\pi}{3}\right)$.

Solution

$$f_x(x, y) = \frac{-yx^{-2}}{1 + \left(\dfrac{y}{x}\right)^2} = \frac{-y}{x^2 + y^2}; \qquad f_x(1, \sqrt{3}) = \frac{-\sqrt{3}}{1+3} = \frac{-\sqrt{3}}{4}$$

$$f_y(x, y) = \frac{x^{-1}}{1 + \left(\dfrac{y}{x}\right)^2} = \frac{x}{x^2 + y^2}; \qquad f_y(1, \sqrt{3}) = \frac{1}{1+3} = \frac{1}{4}$$

The equation of the tangent plane is

$$z - \frac{\pi}{3} = \left(\frac{-\sqrt{3}}{4}\right)(x - 1) + \frac{1}{4}(y - \sqrt{3})$$

or, in standard form,

$$3\sqrt{3}x - 3y + 12z - 4\pi = 0$$

INCREMENTAL APPROXIMATIONS

In Chapter 3, we observed that the tangent line to the curve $y = f(x)$ at the point $P(x_0, y_0)$ is the line that best fits the shape of the curve in the immediate vicinity of P. That is, if f is differentiable at $x = x_0$ and the increment Δx is sufficiently small, then

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\Delta x$$

or, equivalently,

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

Similarly, the tangent plane at $P(x_0, y_0, z_0)$ is the plane that best fits the shape of the surface $z = f(x, y)$ near P, and the analogous **incremental** (or **linear**) approximation formula is as follows.

Incremental Approximation of a Function of Two Variables

If $f(x, y)$ and its partial derivatives f_x and f_y are defined in an open region R containing the point $P(x_0, y_0)$ and f_x and f_y are continuous at P, then

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

so that

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

A graphical interpretation of this incremental approximation formula is shown in Figure 11.21.

The tangent plane to the surface $z = f(x, y)$ has the equation

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

or

$$z - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

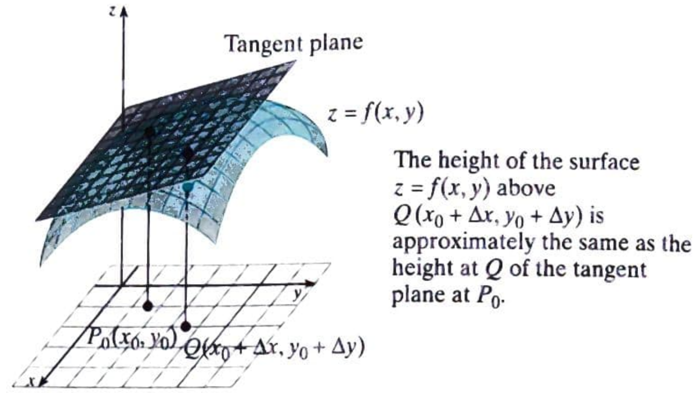


Figure 11.21 Incremental approximation to a function of two variables

As long as we are near (x_0, y_0), the height of the tangent plane is approximately the same as the height of the surface. Thus, if $|\Delta x|$ and $|\Delta y|$ are small, the point $(x_0 + \Delta x, y_0 + \Delta y)$ will be near (x_0, y_0) and we have

$$\underbrace{f(x_0 + \Delta x, y_0 + \Delta y)}_{\substack{\text{Height of } z = f(x,y) \\ \text{above } Q(x_0+\Delta x,\, y_0+\Delta y)}} \approx \underbrace{f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y}_{\text{Height of the tangent plane above } Q}$$

Increments of a function of three variables $f(x, y, z)$ can be defined in a similar fashion. Suppose f has continuous partial derivatives f_x, f_y, f_z in a ball centered at the point (x_0, y_0, z_0). Then if the numbers Δx, Δy, Δz are all sufficiently small, we have

$$\begin{aligned}\Delta f &= f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0) \\ &\approx f_x(x_0, y_0, z_0)\Delta x + f_y(x_0, y_0, z_0)\Delta y + f_z(x_0, y_0, z_0)\Delta z\end{aligned}$$

EXAMPLE 2 Using increments to estimate the change of a function

An open box has length 3 ft, width 1 ft, and height 2 ft and is constructed from material that costs $\$2/\text{ft}^2$ for the sides and $\$3/\text{ft}^2$ for the bottom (see Figure 11.22). Compute the cost of constructing the box, and then use increments to estimate the change in cost if the length and width are each increased by 3 in. and the height is decreased by 4 in.

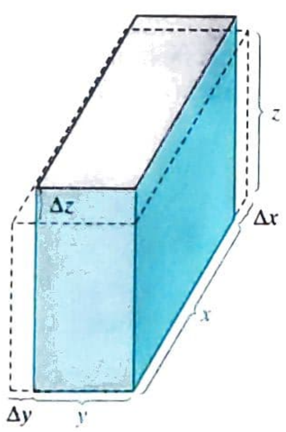


Figure 11.22 Construction of a box

Solution

An open (no top) box with length x, width y, and height z has surface area

$$S = \underbrace{xy}_{\text{Bottom}} + \underbrace{2xz + 2yz}_{\text{Four side faces}}$$

Because the sides cost $\$2/\text{ft}^2$ and the bottom $\$3/\text{ft}^2$, the total cost is

$$C(x, y, z) = 3xy + 2(2xz + 2yz)$$

The partial derivatives of C are

$$C_x = 3y + 4z \qquad C_y = 3x + 4z \qquad C_z = 4x + 4y$$

and the dimensions of the box change by

$$\Delta x = \tfrac{3}{12} = 0.25 \text{ ft} \qquad \Delta y = \tfrac{3}{12} = 0.25 \text{ ft} \qquad \Delta z = \tfrac{-4}{12} \approx -0.33 \text{ ft}$$

Thus, the change in the total cost is approximated by

$$\begin{aligned}\Delta C &\approx C_x(3, 1, 2)\Delta x + C_y(3, 1, 2)\Delta y + C_z(3, 1, 2)\Delta z \\ &= [3(1) + 4(2)](0.25) + [3(3) + 4(2)](0.25) + [4(3) + 4(1)](-\tfrac{4}{12}) \\ &\approx 1.67\end{aligned}$$

That is, the cost increases by approximately \$1.67. ■

EXAMPLE 3 Maximum percentage error using differentials

The radius and height of a right circular cone are measured with errors of at most 3% and 2%, respectively. Use increments to approximate the maximum possible percentage error in computing the volume of the cone using these measurements and the formula $V = \frac{1}{3}\pi R^2 H$.

Solution

We are given that

$$\left|\frac{\Delta R}{R}\right| \le 0.03 \quad \text{and} \quad \left|\frac{\Delta H}{H}\right| \le 0.02$$

The partial derivatives of V are

$$V_R = \tfrac{2}{3}\pi RH \quad \text{and} \quad V_H = \tfrac{1}{3}\pi R^2$$

so the change in V is approximated by

$$\Delta V \approx \left(\tfrac{2}{3}\pi RH\right)\Delta R + \left(\tfrac{1}{3}\pi R^2\right)\Delta H$$

Dividing by the volume $V = \frac{1}{3}\pi R^2 H$, we obtain

$$\frac{\Delta V}{V} \approx \frac{\frac{2}{3}\pi RH\Delta R + \frac{1}{3}\pi R^2 \Delta H}{\frac{1}{3}\pi R^2 H} = 2\left(\frac{\Delta R}{R}\right) + \left(\frac{\Delta H}{H}\right)$$

so that $\left|\frac{\Delta V}{V}\right| \le 2\left|\frac{\Delta R}{R}\right| + \left|\frac{\Delta H}{H}\right| = 2(0.03) + (0.02) = 0.08$. Thus, the maximum percentage error in computing the volume V is approximately 8%. ■

THE TOTAL DIFFERENTIAL

For a function of one variable, $y = f(x)$, we defined the differential dy to be $dy = f'(x)\,dx$. For the two-variable case, we make the following analogous definition.

Total Differential

> The **total differential** of the function $f(x, y)$ is
>
> $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x(x, y)\,dx + f_y(x, y)\,dy$$
>
> where dx and dy are independent variables. Similarly, for a function of three variables $w = f(x, y, z)$ the **total differential** is
>
> $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

EXAMPLE 4 Total differential

Determine the total differential of the given functions:

a. $f(x, y, z) = 2x^3 + 5y^4 - 6z$ **b.** $f(x, y) = x^2 \ln(3y^2 - 2x)$

Solution

a. $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 6x^2\,dx + 20y^3\,dy - 6\,dz$

b. $$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

$$= \left[2x\ln(3y^2 - 2x) + x^2\frac{-2}{3y^2 - 2x}\right]dx + \left[x^2\frac{6y}{3y^2 - 2x}\right]dy$$

$$= \left[2x\ln(3y^2 - 2x) - \frac{2x^2}{3y^2 - 2x}\right]dx + \frac{6x^2y}{3y^2 - 2x}\,dy$$

EXAMPLE 5 Application of the total differential

At a certain factory, the daily output is $Q = 60K^{1/2}L^{1/3}$ units, where K denotes the capital investment (in units of \$1,000) and L the size of the labor force (in worker-hours). The current capital investment is \$900,000, and 1,000 worker-hours of labor are used each day. Estimate the change in output that will result if capital investment is increased by \$1,000 and labor is decreased by 2 worker-hours.

Solution

The change in output is estimated by the total differential dQ. We have $K = 900$, $L = 1{,}000$, $dK = \Delta K = 1$, and $dL = \Delta L = -2$. The total differential of $Q(x, y)$ is

$$\begin{aligned} dQ &= \frac{\partial Q}{\partial K}\,dK + \frac{\partial Q}{\partial L}\,dL \\ &= 60\left(\tfrac{1}{2}\right)K^{-1/2}L^{1/3}\,dK + 60\left(\tfrac{1}{3}\right)K^{1/2}L^{-2/3}\,dL \\ &= 30K^{-1/2}L^{1/3}\,dK + 20K^{1/2}L^{-2/3}\,dL \end{aligned}$$

Substituting for K, L, dK, and dL,

$$dQ = 30(900)^{-1/2}(1{,}000)^{1/3}(1) + 20(900)^{1/2}(1{,}000)^{-2/3}(-2) = -2$$

Thus, the output decreases by approximately 2 units when the capital investment is increased by \$1,000 and labor is decreased by 2 worker-hours.

EXAMPLE 6 Maximum percentage error in an electrical circuit

When two resistances R_1 and R_2 are connected in parallel, the total resistance R satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

If R_1 is measured as 300 ohms with a maximum error of 2% and R_2 is measured as 500 ohms with a maximum error of 3%, what is the maximum percentage error in R?

Solution

We are given that

$$\left|\frac{dR_1}{R_1}\right| \le 0.02 \quad\text{and}\quad \left|\frac{dR_2}{R_2}\right| \le 0.03$$

and we wish to find the maximum value of $\left|\frac{dR}{R}\right|$. Because $R = \frac{R_1R_2}{R_1 + R_2}$ (solve the given equation for R), we have

$$\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1 + R_2)^2} \quad\text{and}\quad \frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1 + R_2)^2} \qquad \text{Quotient rule}$$

it follows that the total differential of R is

$$\begin{aligned} dR &= \frac{\partial R}{\partial R_1}\,dR_1 + \frac{\partial R}{\partial R_2}\,dR_2 \\ &= \frac{R_2^2}{(R_1 + R_2)^2}\,dR_1 + \frac{R_1^2}{(R_1 + R_2)^2}\,dR_2 \end{aligned}$$

We now find $\frac{dR}{R}$ by dividing both sides by R; however, since $\frac{1}{R} = \frac{R_1 + R_2}{R_1 R_2}$, it follows that

$$dR \cdot \frac{1}{R} = \left[\frac{R_2^2}{(R_1 + R_2)^2} dR_1 + \frac{R_1^2}{(R_1 + R_2)^2} dR_2\right] \cdot \frac{R_1 + R_2}{R_1 R_2}$$

$$\frac{dR}{R} = \frac{R_2}{R_1 + R_2} \cdot \frac{dR_1}{R_1} + \frac{R_1}{R_1 + R_2} \cdot \frac{dR_2}{R_2}$$

Finally, apply the triangle inequality (Table 1.1, p. 3) to this relationship:

$$\left|\frac{dR}{R}\right| \le \left|\frac{R_2}{R_1 + R_2}\right|\left|\frac{dR_1}{R_1}\right| + \left|\frac{R_1}{R_1 + R_2}\right|\left|\frac{dR_2}{R_2}\right|$$

$$\le \frac{500}{300 + 500}(0.02) + \frac{300}{300 + 500}(0.03) = 0.02375$$

The maximum percentage is approximately 2.4%. ■

DIFFERENTIABILITY

Recall from Chapter 3 that if $f(x)$ is differentiable at x_0, its increment is

$$\Delta f = f(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x + \epsilon \Delta x$$

where $\epsilon \to 0$ as $\Delta x \to 0$ (see Figure 11.23).

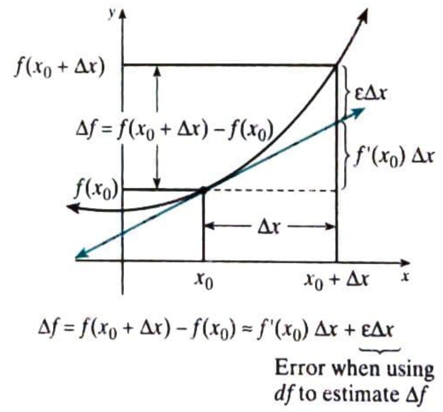


Figure 11.23 Increment of a function f

For a function of two variables, the increment of x is an independent variable denoted by Δx, the increment of y is an independent variable denoted by Δy, and the increment of f at (x_0, y_0) is defined as

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

We use this increment representation to define differentiability as follows.

Definition of Differentiability

The function $f(x, y)$ is **differentiable** at (x_0, y_0) if the increment of f can be expressed as

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as both $\Delta x \to 0$ and $\Delta y \to 0$. In addition, $f(x, y)$ is said to be **differentiable in the region R** of the plane if f is differentiable at each point in R.

In Section 3.1, we showed that a function of one variable is continuous wherever it is differentiable. The following theorem establishes the same result for a function of two variables.

THEOREM 11.2 Differentiability implies continuity

If $f(x, y)$ is differentiable at (x_0, y_0), it is also continuous there.

Proof We wish to show that $f(x, y) \to f(x_0, y_0)$ as $(x, y) \to (x_0, y_0)$ or, equivalently, that

$$\lim_{(x,y)\to(x_0,y_0)} [f(x, y) - f(x_0, y_0)] = 0$$

If we set $\Delta x = x - x_0$ and $\Delta y = y - y_0$ and let Δf denote the increment of f at (x_0, y_0), we have (by substitution)

$$f(x, y) - f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = \Delta f$$

Then, because $(\Delta x, \Delta y) \to (0, 0)$ as $(x, y) \to (x_0, y_0)$, we wish to prove that

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \Delta f = 0$$

Since f is differentiable at (x_0, y_0), we have

$$\Delta f = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. It follows that

$$\begin{aligned}\lim_{(\Delta x,\Delta y)\to(0,0)} \Delta f &= \lim_{(\Delta x,\Delta y)\to(0,0)} [f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y] \\ &= [f_x(x_0, y_0)] \cdot 0 + [f_y(x_0, y_0)] \cdot 0 + 0 + 0 = 0\end{aligned}$$

as required. ❑

WARNING Be careful about how you use the word *differentiable*. In the single-variable case, a function is differentiable at a point if its derivative exists there. However, the word is used differently for a function of two variables. In particular, the existence of the partial derivatives f_x and f_y does not guarantee that the function is differentiable, as illustrated in the following example.

EXAMPLE 7 A nondifferentiable function for which f_x and f_y exist

Let

$$f(x, y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

That is, the function f has the value 1 when (x, y) is in the first quadrant and is 0 elsewhere. Show that the partial derivatives f_x and f_y exist at the origin, but f is not differentiable there.

Solution

Since $f(0, 0) = 0$, we have

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} = 0$$

and similarly, $f_y(0, 0) = 0$. Thus, the partial derivatives both exist at the origin.

If $f(x, y)$ were differentiable at the origin, it would have to be continuous there (Theorem 11.2). Thus, we can show f is not differentiable by showing that it is not continuous at $(0, 0)$. Toward this end, note that $\lim_{(x,y)\to(0,0)} f(x, y)$ is 1 along the line $y = x$ in the first quadrant but is 0 if the approach is along the x-axis. This means that the limit does not exist. Thus, f is not continuous at $(0, 0)$ and consequently is also not differentiable there. ■

Although the existence of partial derivatives at $P(x_0, y_0)$ is not enough to guarantee that $f(x, y)$ is differentiable at P, we do have the following sufficient condition for differentiability.

WARNING Note that the function in Example 7 does not contradict Theorem 11.3 because there is no disk centered at $(0, 0)$ on which f is continuous.

THEOREM 11.3 Sufficient condition for differentiability

If f is a function of x and y, and f, f_x, and f_y are continuous in a disk D centered at (x_0, y_0), then f is differentiable at (x_0, y_0).

Proof The proof is found in advanced calculus texts. ❏

EXAMPLE 8 Establishing differentiability

Show that $f(x, y) = x^2y + xy^3$ is differentiable for all (x, y).

Solution

Compute the partial derivatives

$$f_x(x, y) = \frac{\partial}{\partial x}(x^2y + xy^3) = 2xy + y^3$$

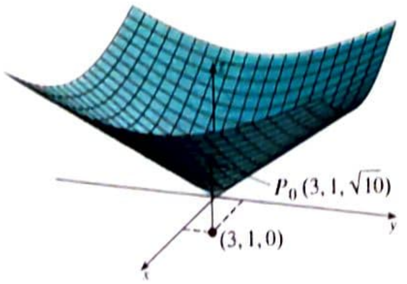
$$f_y(x, y) = \frac{\partial}{\partial y}(x^2y + xy^3) = x^2 + 3xy^2$$

Because f, f_x, and f_y are all polynomials in x and y, they are continuous throughout the plane. Therefore, the sufficient condition for differentiability theorem assures us that f must be differentiable for all x and y. ■

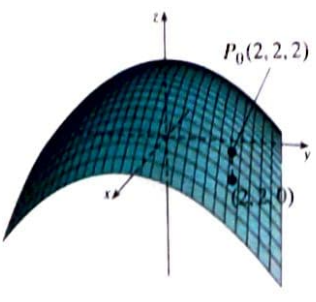
11.4 PROBLEM SET

A *In Problems 1–6, determine the standard-form equations for the tangent plane to the given surface at the prescribed point P_0.*

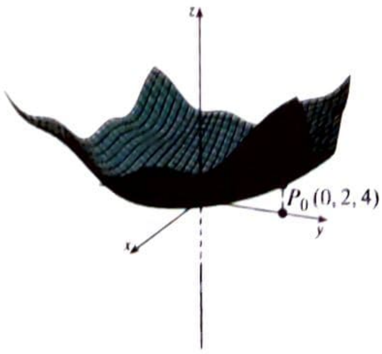
1. $z = \sqrt{x^2 + y^2}$ at $P_0(3, 1, \sqrt{10})$



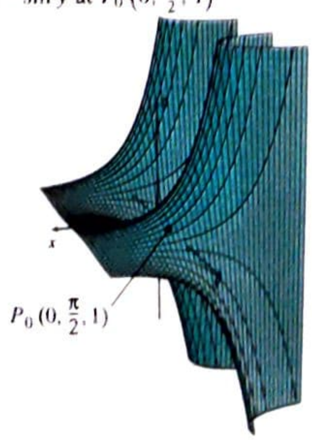
2. $z = 10 - x^2 - y^2$ at $P_0(2, 2, 2)$



3. $f(x, y) = x^2 + y^2 + \sin xy$ at $P_0 = (0, 2, 4)$



4. $f(x, y) = e^{-x} \sin y$ at $P_0\left(0, \frac{\pi}{2}, 1\right)$



40. **Modeling Problem** A business analyst models the sales of a new product by the function

$$Q(x, y) = 20x^{3/2}y$$

where x thousand dollars are spent on development and y thousand dollars on promotion. Current plans call for the expenditure of \$36,000 on development and \$25,000 on promotion. Use the total differential of Q to estimate the change in sales that will result if the amount spent on development is increased by \$500 and the amount spent on promotion is decreased by \$500.

41. Using x hours of skilled labor and y hours of unskilled labor, a manufacturer can produce $f(x, y) = 10xy^{1/2}$ units. Currently, the manufacturer has used 30 h of skilled labor and 36 h of unskilled labor and is planning to use 1 additional hour of skilled labor. Use calculus to estimate the corresponding change that the manufacturer should make in the level of unskilled labor so that the total output will remain the same.

42. **Modeling Problem** A grocer's weekly profit from the sale of two brands of orange juice is modeled by

$$P(x, y) = (x - 30)(70 - 5x + 4y) + (y - 40)(80 + 6x - 7y)$$

dollars, where x cents is the price per can of the first brand and y cents is the price per can of the second. Currently the first brand sells for 50¢ per can and the second for 52¢ per can. Use the total differential to estimate the change in the weekly profit that will result if the grocer raises the price of the first brand by 1¢ per can and lowers the price of the second brand by 2¢ per can.

43. A juice can is 12 cm tall and has a radius of 3 cm. A manufacturer is planning to reduce the height of the can by 0.2 cm and the radius by 0.3 cm. Use a total differential to estimate the percentage decrease in volume that occurs when the new cans are introduced. (Round to the nearest percent.)

44. **Modeling Problem** It is known that the period T of a simple pendulum with small oscillations is modeled by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum and g is the acceleration due to gravity. For a certain pendulum, it is known that $L = 4.03$ ft. It is also known that $g = 32.2$ ft/s^2. What is the approximate error in calculating T by using $L = 4$ and $g = 32$?

45. If the weight of an object that does not float in water is x pounds in the air and its weight in water is y pounds, then the specific gravity of the object is

$$S = \frac{x}{x - y}$$

For a certain object, x and y are measured to be 1.2 lb and 0.5 lb, respectively. It is known that the measuring instrument will not register less than the true weights, but it could register more than the true weights by as much as 0.01 lb. What is the maximum possible error in the computation of the specific gravity?

46. A football has the shape of the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{36} + \frac{z^2}{9} = 1$$

where the dimensions are in inches, and is made of leather 1/8 inch thick. Use differentials to estimate the volume of the leather shell. *Hint*: The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

has volume $V = \frac{4}{3}\pi abc$.

C 47. Show that the following function is not differentiable at $(0, 0)$:

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

48. Compute the total differentials

$$d\left(\frac{x}{x - y}\right) \quad \text{and} \quad d\left(\frac{y}{x - y}\right)$$

Why are these differentials equal?

49. Let A be the area of a triangle with sides a and b separated by an angle θ, as shown in Figure 11.24.

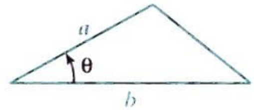


Figure 11.24 Problem 49

Suppose $\theta = \frac{\pi}{6}$, and a is increased by 4% while b is decreased by 3%. Use differentials to estimate the percentage change in A.

50. In Problem 49, suppose that θ also changes by no more than 2%. What is the maximum percentage change in A?

11.5 Chain Rules

IN THIS SECTION chain rule for one parameter, extensions of the chain rule

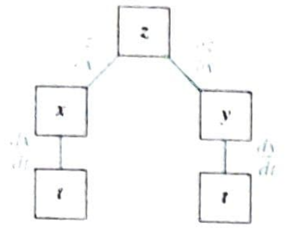
CHAIN RULE FOR ONE PARAMETER

We begin with a differentiable function of two variables $f(x, y)$. If $x = x(t)$ and $y = y(t)$ are, in turn, functions of a single parameter t, then $z = f(x(t), y(t))$ is a composite function of a parameter t. In this case, the chain rule for finding the derivative with respect to one parameter can now be stated.

THEOREM 11.4 The chain rule for one independent parameter

Let $f(x, y)$ be a differentiable function of x and y, and let $x = x(t)$ and $y = y(t)$ be differentiable functions of t. Then $z = f(x, y)$ is a differentiable function of t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$



Tree diagram illustrating the chain rule

➔ **What This Says** The tree diagram shown in the margin is a device for remembering the chain rule. The diagram begins at the top with the dependent variable z and cascades downward in two branches, first to the independent variables x and y, and then to the parameter t on which they each depend. Each branch segment is labeled with a derivative, and the chain rule is obtained by first multiplying the derivatives on the two segments of each branch and then adding to obtain

$$\frac{dz}{dt} = \underbrace{\frac{\partial z}{\partial x}\frac{dx}{dt}}_{\textit{left branch}} + \underbrace{\frac{\partial z}{\partial y}\frac{dy}{dt}}_{\textit{right branch}}$$

Proof Recall that because $z = f(x, y)$ is differentiable, we can write the increment Δz in the following form:

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as both $\Delta x \to 0$ and $\Delta y \to 0$. Dividing by $\Delta t \neq 0$, we obtain

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t}$$

Because x and y are functions of t, we can write their increments as

$$\Delta x = x(t + \Delta t) - x(t) \quad \text{and} \quad \Delta y = y(t + \Delta t) - y(t)$$

We know that x and y both vary continuously with t (remember, they are differentiable), and it follows that $\Delta x \to 0$ and $\Delta y \to 0$ as $\Delta t \to 0$, so that $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta t \to 0$. Therefore, we have

$$\begin{aligned}\frac{dz}{dt} &= \lim_{\Delta t \to 0}\frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0}\left[\frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t}\right]\\ &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + 0\frac{dx}{dt} + 0\frac{dy}{dt}\end{aligned}$$ ❑

EXAMPLE 1 Verifying the chain rule explicitly

Let $z = x^2 + y^2$, where $x = \dfrac{1}{t}$ and $y = t^2$. Compute $\dfrac{dz}{dt}$ in two ways:

a. by first expressing z explicitly in terms of t
b. by using the chain rule

Solution

a. By substituting $x = \frac{1}{t}$ and $y = t^2$, we find that

$$z = x^2 + y^2 = \left(\frac{1}{t}\right)^2 + (t^2)^2 = t^{-2} + t^4 \qquad \text{for } t \neq 0$$

Thus, $\frac{dz}{dt} = -2t^{-3} + 4t^3$.

b. Because $z = x^2 + y^2$ and $x = t^{-1}$, $y = t^2$,

$$\frac{\partial z}{\partial x} = 2x; \qquad \frac{\partial z}{\partial y} = 2y; \qquad \frac{dx}{dt} = -t^{-2}; \qquad \frac{dy}{dt} = 2t$$

Use the chain rule for one independent parameter:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (2x)(-t^{-2}) + 2y(2t) && \text{Chain rule} \\ &= 2(t^{-1})(-t^{-2}) + 2(t^2)(2t) && \text{Substitute} \\ &= -2t^{-3} + 4t^3 \end{aligned}$$

■

EXAMPLE 2 Chain rule for one independent parameter

Let $z = \sqrt{x^2 + 2xy}$, where $x = \cos\theta$ and $y = \sin\theta$. Find $\frac{dz}{d\theta}$.

Solution

$$\frac{\partial z}{\partial x} = \tfrac{1}{2}(x^2 + 2xy)^{-1/2}(2x + 2y) \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \tfrac{1}{2}(x^2 + 2xy)^{-1/2}(2x)$$

Also, $\frac{dx}{d\theta} = -\sin\theta$ and $\frac{dy}{d\theta} = \cos\theta$. Use the chain rule for one independent parameter to find

$$\begin{aligned} \frac{dz}{d\theta} &= \frac{\partial z}{\partial x}\frac{dx}{d\theta} + \frac{\partial z}{\partial y}\frac{dy}{d\theta} \\ &= \tfrac{1}{2}(x^2 + 2xy)^{-1/2}(2x + 2y)(-\sin\theta) + \tfrac{1}{2}(x^2 + 2xy)^{-1/2}(2x)(\cos\theta) \\ &= (x^2 + 2xy)^{-1/2}(x\cos\theta - x\sin\theta - y\sin\theta) \end{aligned}$$

■

EXAMPLE 3 Related rate application using the chain rule

A right circular cylinder (see Figure 11.25) is changing in such a way that its radius r is increasing at the rate of 3 in./min and its height h is decreasing at the rate of 5 in./min. At what rate is the volume of the cylinder changing when the radius is 10 in. and the height is 8 in.?

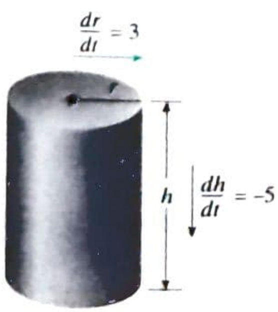


Figure 11.25 Right circular cylinder

Solution

The volume of the cylinder is $V = \pi r^2 h$, and we are given $\frac{dr}{dt} = 3$ and $\frac{dh}{dt} = -5$. We find that

$$\frac{\partial V}{\partial r} = \pi(2r)h \quad \text{and} \quad \frac{\partial V}{\partial h} = \pi r^2(1)$$

By the chain rule for one parameter,

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = 2\pi r h\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$$

Thus, at the instant when $r = 10$ and $h = 8$, we have

$$\frac{dV}{dt} = 2\pi(10)(8)(3) + \pi(10)^2(-5) = -20\pi$$

The volume is decreasing at the rate of about 62.8 in.^{3}/min.

■

If $F(x, y) = 0$ defines y implicitly as a differentiable function x, we can regard x as a parameter and apply the chain rule to obtain

$$0 = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

so

$$\frac{dy}{dx} = \frac{-\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \qquad \text{provided } F_y \neq 0$$

This formula provides a useful alternative to the procedure for implicit differentiation introduced in Section 3.6. This alternative procedure is illustrated in the following example.

EXAMPLE 4 Implicit differentiation using partial derivatives

If y is a differentiable function of x such that

$$\sin(x+y) + \cos(x-y) = y$$

find $\dfrac{dy}{dx}$.

Solution

Let $F(x, y) = \sin(x+y) + \cos(x-y) - y$, so that $F(x, y) = 0$. Then

$$F_x = \cos(x+y) - \sin(x-y)$$
$$F_y = \cos(x+y) - \sin(x-y)(-1) - 1$$

so

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-[\cos(x+y) - \sin(x-y)]}{\cos(x+y) + \sin(x-y) - 1}$$

When z is defined implicitly in terms of x and y by an equation $F(x, y, z) = 0$, the chain rule can be used to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in terms of F_x, F_y, and F_z. The procedure is outlined in Problem 57.

EXAMPLE 5 Second derivative of a function of two variables

Let $z = f(x, y)$, where $x = at$ and $y = bt$ for constants a and b. Assuming all necessary differentiability, find d^2z/dt^2 in terms of the partial derivatives of z.

Solution

By using the chain rule, we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

and

$$\begin{aligned}
\frac{d^2z}{dt^2} &= \frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{d}{dt}\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right) \\
&= \frac{\partial z}{\partial x}\left[\frac{d}{dt}\left(\frac{dx}{dt}\right)\right] + \left[\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right)\right]\frac{dx}{dt} + \frac{\partial z}{\partial y}\left[\frac{d}{dt}\left(\frac{dy}{dt}\right)\right] + \left[\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right)\right]\frac{dy}{dt} \\
&= \left[\frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{dx}{dt}\left(\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial x \partial y}\frac{dy}{dt}\right)\right] + \left[\frac{\partial z}{\partial y}\frac{d^2y}{dt^2} + \frac{dy}{dt}\left(\frac{\partial^2 z}{\partial y \partial x}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}\right)\right]
\end{aligned}$$

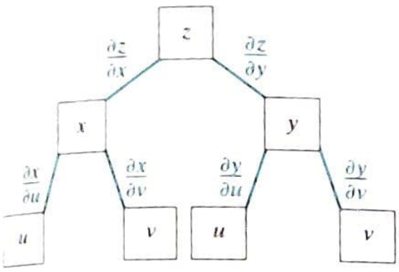
Substituting $\dfrac{dx}{dt} = a$ and $\dfrac{dy}{dt} = b$, we obtain

$$\frac{d^2z}{dt^2} = \left[\frac{\partial z}{\partial x}(0) + a\left(\frac{\partial^2 z}{\partial x^2}a + \frac{\partial^2 z}{\partial x \partial y}b\right)\right] + \left[\frac{\partial z}{\partial y}(0) + b\left(\frac{\partial^2 z}{\partial y \partial x}a + \frac{\partial^2 z}{\partial y^2}b\right)\right]$$

$$= a^2\frac{\partial^2 z}{\partial x^2} + 2ab\frac{\partial^2 z}{\partial x \partial y} + b^2\frac{\partial^2 z}{\partial y^2} \qquad \text{Note } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

■

EXTENSIONS OF THE CHAIN RULE



Next, we will consider the kind of composite function that occurs when x and y are both functions of two parameters. Specifically, let $z = F(x, y)$, where $x = x(u, v)$ and $y = y(u, v)$ are both functions of two independent parameters u and v. Then $z = F[x(u, v), y(u, v)]$ is a composite function of u and v, and with suitable assumptions regarding differentiability, we can find the partial derivatives $\partial z/\partial u$ and $\partial z/\partial v$ by applying the chain rule obtained in the following theorem.

THEOREM 11.5 The chain rule for two independent parameters

Suppose $z = f(x, y)$ is differentiable at (x, y) and that the partial derivatives of $x = x(u, v)$ and $y = y(u, v)$ exist at (u, v). Then the composite function $z = f[x(u, v), y(u, v)]$ is differentiable at (u, v) with

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Proof This version of the chain rule follows immediately from the chain rule for one independent parameter. For example, if v is fixed, the composite function $z = f[x(u, v), y(u, v)]$ depends on u alone, and we have the situation described in the chain rule of one independent variable. We apply this chain rule with a partial derivative (because x and y are functions of more than one variable):

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

The formula for $\dfrac{\partial z}{\partial v}$ can be established in a similar fashion. ❑

EXAMPLE 6 Chain rule for two independent parameters

Let $z = 4x - y^2$, where $x = uv^2$ and $y = u^3v$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Solution

First find the partial derivatives:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(4x - y^2) = 4 \qquad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(4x - y^2) = -2y$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(uv^2) = v^2 \qquad \frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(u^3v) = 3u^2v$$

and

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(uv^2) = 2uv \qquad \frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(u^3v) = u^3$$

Therefore, the chain rule for two independent parameters gives

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= (4)(v^2) + (-2y)(3u^2v) = 4v^2 - 2(u^3v)(3u^2v) = 4v^2 - 6u^5v^2$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$
$$= (4)(2uv) + (-2y)(u^3) = 8uv - 2(u^3v)u^3 = 8uv - 2u^6v$$

EXAMPLE 7 Implicit differentiation using the chain rule

If f is differentiable and $z = u + f(u^2v^2)$, show that $u\dfrac{\partial z}{\partial u} - v\dfrac{\partial z}{\partial v} = u$.

Solution

Let $w = u^2v^2$, so $z = u + f(w)$. Then, according to the chain rule,

$$\frac{\partial z}{\partial u} = 1 + \frac{df}{dw}\frac{\partial w}{\partial u} \qquad \frac{\partial z}{\partial v} = \frac{df}{dw}\frac{\partial w}{\partial v}$$
$$= 1 + f'(w)(2uv^2) \qquad = f'(w)(2u^2v)$$

so that

$$u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = u\left[1 + f'(w)(2uv^2)\right] - v\left[f'(w)(2u^2v)\right]$$
$$= u + f'(w)\left[u(2uv^2) - v(2u^2v)\right]$$
$$= u$$

The chain rules can be extended to functions of three or more variables. For instance, if $w = f(x, y, z)$ is a differentiable function of three variables and $x = x(t)$, $y = y(t)$, $z = z(t)$ are each differentiable functions of t, then w is a differentiable composite function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

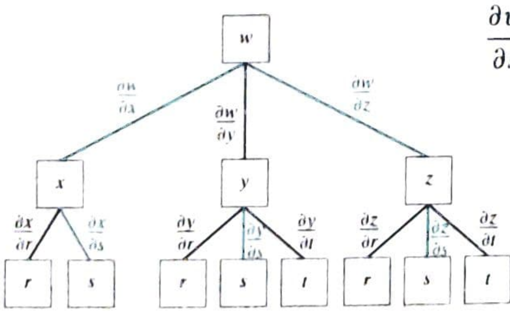
In general, if $w = f(x_1, x_2, \ldots, x_n)$ is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$, which in turn are differentiable functions of m parameters $t_1, t_2, \ldots, t_m$, then

$$\frac{\partial w}{\partial t_1} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$
$$\vdots$$
$$\frac{\partial w}{\partial t_m} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

EXAMPLE 8 Chain rule for a function of three variables with three parameters

Find $\dfrac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3$, where $x = e^{rs^2}$, $y = \ln\dfrac{r+s}{t}$, and $z = rst^2$.

Solution



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$
$$= \left[\frac{\partial}{\partial x}(4x + y^2 + z^3)\right]\left[\frac{\partial}{\partial s}(e^{rs^2})\right] + \left[\frac{\partial}{\partial y}(4x + y^2 + z^3)\right]\left[\frac{\partial}{\partial s}\left(\ln\frac{r+s}{t}\right)\right]$$
$$+ \left[\frac{\partial}{\partial z}(4x + y^2 + z^3)\right]\left[\frac{\partial}{\partial s}(rst^2)\right]$$
$$= 4\left[e^{rs^2}(2rs)\right] + 2y\left(\frac{1}{\frac{r+s}{t}}\right)\left(\frac{1}{t}\right) + 3z^2(rt^2)$$
$$= 8rse^{rs^2} + \frac{2y}{r+s} + 3rt^2z^2$$

In terms of r, s, and t, the partial derivative is

$$\frac{\partial w}{\partial s} = 8rse^{rs^2} + \frac{2}{r+s}\ln\frac{r+s}{t} + 3r^3s^2t^6$$

11.5 PROBLEM SET

1. **WHAT DOES THIS SAY?** Discuss the various chain rules and the need for such chain rules.
2. **WHAT DOES THIS SAY?** Discuss the usefulness of the schematic (tree) representation for the chain rules.
3. **WHAT DOES THIS SAY?** Write out a chain rule for a function of two variables and three independent parameters.

In Problems 4–7, the function $z = f(x, y)$ depends on x and y, which in turn are each functions of t. In each case, find dz/dt in two different ways:

a. Express z explicitly in terms of t.

b. Use the chain rule for one parameter.

4. $f(x, y) = 2xy + y^2$, where $x = -3t^2$ and $y = 1 + t^3$
5. $f(x, y) = (4 + y^2)x$, where $x = e^{2t}$ and $y = e^{3t}$
6. $f(x, y) = (1 + x^2 + y^2)^{1/2}$, where $x = \cos 5t$ and $y = \sin 5t$
7. $f(x, y) = xy^2$, where $x = \cos 3t$ and $y = \tan 3t$

In Problems 8–11, the function $F(x, y)$ depends on x and y. Let $x = x(u, v)$ and $y = y(u, v)$ be given functions of u and v. Let $z = F[x(u, v), y(u, v)]$ and find the partial derivatives $\partial z/\partial u$ and $\partial z/\partial v$ in these two ways:

a. Express z explicitly in terms of u and v.

b. Apply the chain rule for two independent parameters.

8. $F(x, y) = x + y^2$, where $x = u + v$ and $y = u - v$
9. $F(x, y) = x^2 + y^2$, where $x = u \sin v$ and $y = u - 2v$
10. $F(x, y) = e^{xy}$, where $x = u - v$ and $y = u + v$
11. $F(x, y) = \ln xy$, where $x = e^{uv^2}$, $y = e^{uv}$

Write out the chain rule for the functions given in Problems 12–15.

12. $z = f(x, y)$, where $x = x(s, t)$, $y = y(s, t)$
13. $w = f(x, y, z)$, where $x = x(s, t)$, $y = y(s, t)$, $z = z(s, t)$
14. $t = f(u, v)$, where $u = u(x, y, z, w)$, $v = v(x, y, z, w)$
15. $w = f(x, y, z)$, where $x = x(s, t, u)$, $y = y(s, t, u)$, $z = z(s, t, u)$

Find the indicated derivatives or partial derivatives in Problems 16–21. Leave your answers in mixed form (x, y, z, t).

16. Find $\dfrac{dw}{dt}$, where $w = \ln(x + 2y - z^2)$ and $x = 2t - 1$, $y = \dfrac{1}{t}$, $z = \sqrt{t}$.
17. Find $\dfrac{dw}{dt}$, where $w = \sin xyz$ and $x = 1 - 3t$, $y = e^{1-t}$, $z = 4t$.
18. Find $\dfrac{dw}{dt}$, where $w = ze^{xy^2}$ and $x = \sin t$, $y = \cos t$, $z = \tan 2t$.
19. Find $\dfrac{dw}{dt}$, where $w = e^{x^3+yz}$ and $x = \dfrac{2}{t}$, $y = \ln(2t - 3)$, $z = t^2$.
20. Find $\dfrac{\partial w}{\partial r}$, where $w = e^{2x-y+3z^2}$ and $x = r+s-t$, $y = 2r-3s$, $z = \cos rst$.
21. Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial t}$, where $w = \dfrac{x+y}{2-z}$ and $x = 2rs$, $y = \sin rt$, $z = st^2$

In Problems 22–27, assume the given equations define y as a differentiable function of x and find dy/dx using the procedure illustrated in Example 4.

22. $x^2y + \sqrt{xy} = 4$
23. $(x^2 - y)^{3/2} + x^2y = 2$
24. $x^2y + \ln(2x + y) = 5$
25. $x \cos y + y \tan^{-1} x = x$
26. $xe^{xy} + ye^{-xy} = 3$
27. $\tan^{-1}\left(\dfrac{x}{y}\right) = \tan^{-1}\left(\dfrac{y}{x}\right)$

Find the following higher-order partial derivatives in Problems 28–33.

a. $\dfrac{\partial^2 z}{\partial x \partial y}$ *b.* $\dfrac{\partial^2 z}{\partial x^2}$ *c.* $\dfrac{\partial^2 z}{\partial y^2}$

28. $x^3 + y^2 + z^2 = 5$
29. $xyz = 2$
30. $\ln(x + y) = y^2 + z$
31. $x^{-1} + y^{-1} + z^{-1} = 3$
32. $x \cos y = y + z$
33. $z^2 + \sin x = \tan y$
34. Let $f(x, y)$ be a differentiable function of x and y, and let $x = r\cos\theta$, $y = r\sin\theta$ for $r > 0$ and $0 < \theta < 2\pi$.
    a. If $z = f[x(r, \theta), y(r, \theta)]$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$.
    b. Show that
    $$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$
35. Let $z = f(x, y)$, where $x = au$ and $y = bv$, with a, b constants. Express $\partial^2 z/\partial u^2$ and $\partial^2 z/\partial v^2$ in terms of the partial derivatives of z with respect to x and y. Assume the existence and continuity of all necessary first and second partial derivatives.
36. Let (x, y, z) lie on the ellipsoid
    $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
    Without solving for z explicitly in terms of x and y, compute the higher-order partial derivatives
    $$\frac{\partial^2 z}{\partial x^2} \quad\text{and}\quad \frac{\partial^2 z}{\partial x \partial y}$$
37. The dimensions of a rectangular box are linear functions of time, $\ell(t)$, $w(t)$, and $h(t)$. If the length and width are increasing at 2 in./sec and the height is decreasing at 3 in./sec, find the rates at which the volume V and the surface area S are changing with respect to time. If $\ell(0) = 10$, $w(0) = 8$, and $h(0) = 20$, is V increasing or decreasing when $t = 5$ sec? What about S when $t = 5$?

54. If $f(u, v, w)$ is differentiable and $u = x - y$, $v = y - z$, and $w = z - x$, what is

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}?$$

55. The *Cauchy-Riemann equations* are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(See Problem 51, Section 11.3.) Show that if x and y are expressed in terms of polar coordinates, the Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

56. Let $T(x, y)$ be the temperature at each point (x, y) in a portion of the plane that contains the ellipse $x = 2\cos t$, $y = \sin t$ for $0 \leq t \leq 2\pi$. Suppose

$$\frac{\partial T}{\partial x} = y \quad \text{and} \quad \frac{\partial T}{\partial y} = x$$

a. Find $\dfrac{dT}{dt}$ and $\dfrac{d^2T}{dt^2}$ by using the chain rule.

b. Locate the maximum and minimum temperatures on the ellipse.

57. Let $F(x, y, z)$ be a function of three variables with continuous partial derivatives F_x, F_y, F_z in a certain region where $F(x, y, z) = C$ for some constant C. Use the chain rule for two parameters and the fact that x and y are independent variables to show that (for $F_z \neq 0$)

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Use these formulas to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if z is defined implicitly by the equation $x^2 + 2xyz + y^3 + e^z = 4$.

58. Suppose the system

$$\begin{cases} xu + yv - uv = 0 \\ yu - xv + uv = 0 \end{cases}$$

can be solved for u and v in terms of x and y, so that $u = u(x, y)$ and $v = v(x, y)$. Use implicit differentiation to find the partial derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$.

59. A function $f(x, y)$ is said to be *homogeneous of degree n* if

$$f(tx, ty) = t^n f(x, y) \qquad \text{for all } t > 0$$

a. Show that $f(x, y) = x^2y + 2y^3$ is homogeneous and find its degree.

b. If $f(x, y)$ is homogeneous of degree n, show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

60. Suppose that F and G are functions of three variables and that it is possible to solve the equations $F(x, y, z) = 0$ and $G(x, y, z) = 0$ for y and z in terms of x, so that $y = y(x)$ and $z = z(x)$. Use the chain rule to express dy/dx and dz/dx in terms of the partial derivatives of F and G. Assume these partials are continuous and that

$$\frac{\partial F}{\partial y}\frac{\partial G}{\partial z} \neq \frac{\partial F}{\partial z}\frac{\partial G}{\partial y}$$

11.6 Directional Derivatives and the Gradient

IN THIS SECTION the directional derivative, the gradient, maximal property of the gradient, functions of three variables, normal property of the gradient, tangent planes and normal lines

Suppose $z = T(x, y)$ gives the temperature at each point (x, y) in a region R of the plane, and let $P_0(x_0, y_0)$ be a particular point in R. Then we know that the partial derivative $T_x(x_0, y_0)$ gives the rate at which the temperature changes for a move from P_0 in the x-direction, while the rate of temperature change in the y-direction is given by $T_y(x_0, y_0)$. Suppose we want to find the direction of greatest temperature change, which may be in a direction not parallel to either coordinate axes. To answer this question, we will introduce the concept of *directional derivative* in this section and examine its properties.

THE DIRECTIONAL DERIVATIVE

In Chapter 3 we defined the *slope of a curve* at a point to be the ratio of the change in the dependent variable to the change in the independent variable at the given point. To determine the slope of the tangent line at a point $P_0(x_0, y_0)$ on a surface defined by $z = f(x, y)$, we need to specify the *direction* in which we wish to measure. We do this by using vectors. In Section 11.3 we found the slope parallel to the xz-plane to be the partial derivative $f_x(x_0, y_0)$. We could have specified this direction in terms of the unit vector $\mathbf{i}$ (x-direction), while $f_y(x, y)$ could have been specified in terms of the unit vector $\mathbf{j}$. Finally, to measure the slope of the tangent line in an *arbitrary* direction, we use a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ in that direction.

To find the desired slope, we look at the intersection of the surface with the vertical plane passing through the point P_0 parallel to the vector $\mathbf{u}$, as shown in Figure 11.26. This vertical plane intersects the surface to form a curve C, and we define the slope of the surface at P_0 in the direction of $\mathbf{u}$ to be the slope of the tangent line to the curve C defined by $\mathbf{u}$ at that point.

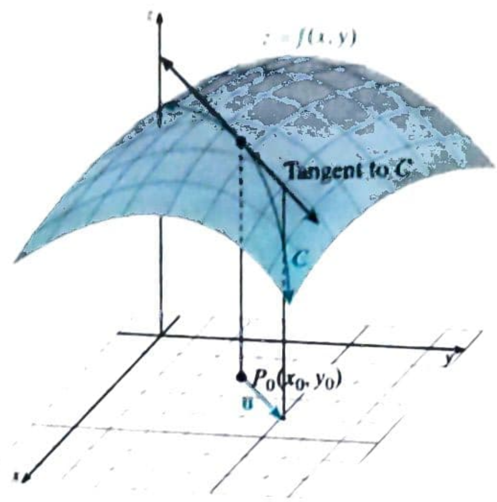


Figure 11.26 The directional derivative $D_{\mathbf{u}} f(x_0, y_0)$ is the slope of the tangent line to the curve on the surface $z = f(x, y)$ in the direction of the unit vector $\mathbf{u}$ at $P_0(x_0, y_0)$.

We summarize this idea of slope *in a particular direction* with the following definition.

Directional Derivative

WARNING Remember, $\mathbf{u}$ must be a *unit* vector.

Let f be a function of two variables, and let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ be a unit vector. The **directional derivative of f at $P_0(x_0, y_0)$** in the direction of $\mathbf{u}$ is given by

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

provided the limit exists.

At a particular point $P_0(x_0, y_0)$, there are infinitely many directional derivatives to the graph of $z = f(x, y)$, one for each direction radiating from P_0. Two of these are the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$. To see this, note that if $\mathbf{u} = \mathbf{i}$ (so $u_1 = 1$ and $u_2 = 0$), then

$$D_{\mathbf{i}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = f_x(x_0, y_0)$$

and if $\mathbf{u} = \mathbf{j}$ (so $u_1 = 0$ and $u_2 = 1$),

$$D_{\mathbf{j}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} = f_y(x_0, y_0)$$

The definition of the directional derivative is similar to the definition of the derivative of a function of a single variable. Just as with a single variable, it is difficult to apply the definition directly. Fortunately, the following theorem allows us to find directional derivatives more efficiently than by using the definition.

THEOREM 11.6 Directional derivatives using partial derivatives

Let $f(x, y)$ be a function that is differentiable at $P_0(x_0, y_0)$. Then f has a directional derivative in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ given by

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

Proof We define a function F of a single variable h by $F(h) = f(x_0+hu_1, y_0+hu_2)$, so that

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

$$= \lim_{h\to 0} \frac{F(h) - F(0)}{h} = F'(0)$$

Apply the chain rule with $x = x_0 + hu_1$ and $y = y_0 + hu_2$:

$$F'(h) = \frac{dF}{dh} = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)u_1 + f_y(x, y)u_2$$

When $h = 0$, we have $x = x_0$ and $y = y_0$, so that

$$D_{\mathbf{u}} f(x_0, y_0) = F'(0) = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \qquad \square$$

EXAMPLE 1 Finding a directional derivative using partial derivatives

Find the directional derivative of $f(x, y) = 3 - 2x^2 + y^3$ at the point $P(1, 2)$ in the direction of the unit vector $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$.

Solution

First, find the partial derivatives $f_x(x, y) = -4x$ and $f_y(x, y) = 3y^2$. Then since $u_1 = \frac{1}{2}$ and $u_2 = -\frac{\sqrt{3}}{2}$, we have

$$D_{\mathbf{u}} f(1, 2) = f_x(1, 2)\left(\frac{1}{2}\right) + f_y(1, 2)\left(-\frac{\sqrt{3}}{2}\right)$$

$$= -4(1)\left(\frac{1}{2}\right) + 3(2)^2\left(-\frac{\sqrt{3}}{2}\right) = -2 - 6\sqrt{3} \approx -12.4 \qquad \blacksquare$$

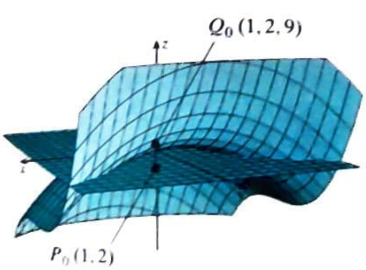


Figure 11.27 The graph of the surface $z = 3 - 2x^2 + y^3$

The directional derivative of $f(x, y)$ at the point $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2\rangle$ can be interpreted as both a rate of change and a slope. For instance, in Example 1, the intersection of the surface $z = 3 - 2x^2 + y^3$ with the vertical plane through the point $P(1, 2)$ parallel to the unit vector $\mathbf{u} = \left\langle \frac{1}{2}, \frac{-\sqrt{3}}{2} \right\rangle$ is a curve C, and the directional derivative $D_{\mathbf{u}} f(1, 2) = -12.3$ is the slope of C at the point $Q(1, 2, 9)$ on the surface above P, as shown in Figure 11.27. The directional derivative also gives the rate at which the function $f(x, y) = 3 - 2x^2 + y^3$ changes as a point (x, y) moves from P in the direction of $\mathbf{u}$.

THE GRADIENT

The directional derivative $D_{\mathbf{u}} f(x, y)$ can be expressed concisely in terms of a vector function called the *gradient*, which has many important uses in mathematics. The gradient of a function of two variables may be defined as follows.

Gradient

Let f be a differentiable function at (x, y) and let $f(x, y)$ have partial derivatives $f_x(x, y)$ and $f_y(x, y)$. Then the **gradient** of f, denoted by ∇f (pronounced "del eff"), is a vector given by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

The value of the gradient at the point $P_0(x_0, y_0)$ is denoted by

$$\nabla f_0 = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$$

Note: Think of the symbol ∇ as an "operator" on a function that produces a vector. Another notation for ∇f is **grad** $f(x, y)$.

EXAMPLE 2 Finding the gradient of a given function

Find $\nabla f(x, y)$ for $f(x, y) = x^2y + y^3$.

Solution

Begin with the partial derivatives:

$$f_x(x, y) = \frac{\partial}{\partial x}(x^2y + y^3) = 2xy \quad \text{and} \quad f_y(x, y) = \frac{\partial}{\partial y}(x^2y + y^3) = x^2 + 3y^2$$

Then

$$\nabla f(x, y) = 2xy\mathbf{i} + (x^2 + 3y^2)\mathbf{j}$$

The following theorem shows how the directional derivative can be expressed in terms of the gradient.

THEOREM 11.7 The gradient formula for the directional derivative

If f is a differentiable function of x and y, then the directional derivative of f at the point $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f_0 \cdot \mathbf{u}$$

Proof Because $\nabla f_0 = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$ and $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, we have

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f_0 \cdot \mathbf{u} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

EXAMPLE 3 Using the gradient formula to compute a directional derivative

Find the directional derivative of $f(x, y) = \ln(x^2 + y^3)$ at $P_0(1, -3)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$.

Solution

$$f_x(x, y) = \frac{2x}{x^2 + y^3}, \quad \text{so} \quad f_x(1, -3) = -\frac{2}{26}$$

$$f_y(x, y) = \frac{3y^2}{x^2 + y^3}, \quad \text{so} \quad f_y(1, -3) = -\frac{27}{26}$$

$$\nabla f_0 = \nabla f(1, -3) = -\tfrac{2}{26}\mathbf{i} - \tfrac{27}{26}\mathbf{j}$$

A unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2\mathbf{i} - 3\mathbf{j}}{\sqrt{2^2 + (-3)^2}} = \frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{j})$$

Thus,

$$D_{\mathbf{u}}(x, y) = \nabla f \cdot \mathbf{u} = \left(-\frac{2}{26}\right)\left(\frac{2}{\sqrt{13}}\right) + \left(-\frac{27}{26}\right)\left(-\frac{3}{\sqrt{13}}\right)$$

$$= \frac{77\sqrt{13}}{338}$$

Although a differentiable function of one variable $f(x)$ has exactly one derivative $f'(x)$, a differentiable function of two variables $F(x, y)$ has two partial derivatives and an infinite number of directional derivatives. Is there any single mathematical concept for functions of several variables that is the analogue of the derivative of a function of a single variable? The properties listed in the following theorem suggest that the gradient plays this role.

THEOREM 11.8 Basic properties of the gradient

Let f and g be differentiable functions. Then

Constant rule	$\nabla c = \mathbf{0}$ for any constant c
Linearity rule	$\nabla(af + bg) = a\nabla f + b\nabla g$ for constants a and b
Product rule	$\nabla(fg) = f\nabla g + g\nabla f$
Quotient rule	$\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}, \quad g \neq 0$
Power rule	$\nabla(f^n) = nf^{n-1}\nabla f$

Proof **Linearity rule**

$$\begin{aligned}\nabla(af + bg) &= (af + bg)_x\mathbf{i} + (af + bg)_y\mathbf{j} = (af_x + bg_x)\mathbf{i} + (af_y + bg_y)\mathbf{j}\\ &= af_x\mathbf{i} + bg_x\mathbf{i} + af_y\mathbf{j} + bg_y\mathbf{j} = a(f_x\mathbf{i} + f_y\mathbf{j}) + b(g_x\mathbf{i} + g_y\mathbf{j})\\ &= a\nabla f + b\nabla g\end{aligned}$$

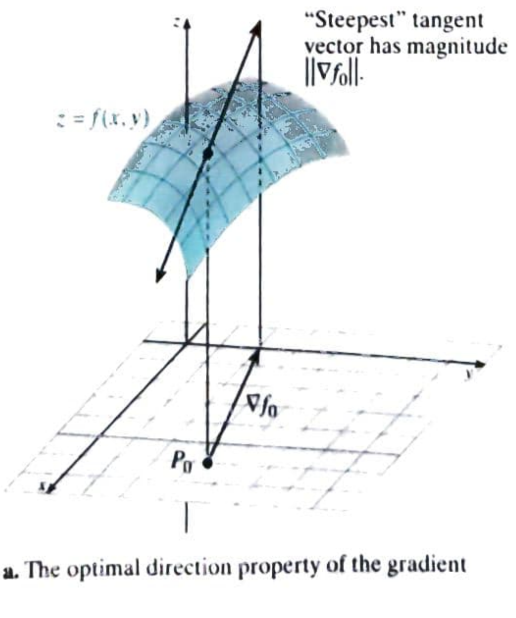
Power rule

$$\begin{aligned}\nabla f^n &= [f^n]_x\mathbf{i} + [f^n]_y\mathbf{j} = nf^{n-1}f_x\mathbf{i} + nf^{n-1}f_y\mathbf{j}\\ &= nf^{n-1}[f_x\mathbf{i} + f_y\mathbf{j}] = nf^{n-1}\nabla f\end{aligned}$$

The other rules are left for the problem set (Problem 57). ❑

MAXIMAL PROPERTY OF THE GRADIENT

In applications, it is often useful to compute the greatest rate of increase (or decrease) of a given function at a specified point. The direction in which this occurs is called the direction of **steepest ascent** (or **steepest descent**). For example, suppose the function $z = f(x, y)$ gives the altitude of a skier coming down a slope, and we want to state a theorem that will give the skier the *compass direction* of the path of steepest descent (see Figure 11.28b). We emphasize the words "compass direction" because the gradient gives direction in the xy-plane and does not itself point up or down the mountain. The following theorem shows how the direction of maximum change is determined by the gradient (see Figure 11.28).



a. The optimal direction property of the gradient


b. Skier on a slope

Figure 11.28 Steepest ascent or steepest descent

THEOREM 11.9 Maximal direction property of the gradient

Suppose f is differentiable at the point P_0 and that the gradient of f at P_0 satisfies $\nabla f_0 \neq \mathbf{0}$. Then

a. The largest value of the directional derivative $D_{\mathbf{u}}f$ at P_0 is $\|\nabla f_0\|$ and occurs when the unit vector $\mathbf{u}$ points in the direction of ∇f_0.

b. The smallest value of $D_{\mathbf{u}}f$ at P_0 is $-\|\nabla f_0\|$ and occurs when $\mathbf{u}$ points in the direction of $-\nabla f_0$.

Proof If $\mathbf{u}$ is any unit vector, then

$$D_{\mathbf{u}}f = \nabla f_0 \cdot \mathbf{u} = \|\nabla f_0\| \, (\|\mathbf{u}\| \cos\theta) = \|\nabla f_0\| \cos\theta$$

where θ is the angle between ∇f_0 and $\mathbf{u}$. But $\cos\theta$ assumes its largest value 1 at $\theta = 0$; that is, when $\mathbf{u}$ points in the direction ∇f_0. Thus, the largest possible value of $D_{\mathbf{u}}f$ is

$$D_{\mathbf{u}}f = \|\nabla f_0\|(1) = \|\nabla f_0\|$$

Statement **b** may be established in a similar fashion by noting that $\cos\theta$ assumes its smallest value -1 when $\theta = \pi$. This value occurs when $\mathbf{u}$ points toward $-\nabla f_0$, and in this direction

$$D_{\mathbf{u}}f = \|\nabla f_0\|(-1) = -\|\nabla f_0\|$$

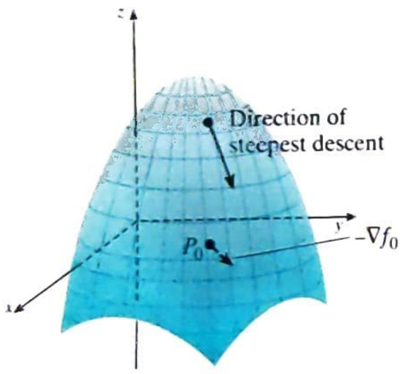


Figure 11.29 The direction of steepest descent

➜ **What This Says** The theorem states that at P_0 the function f increases most rapidly in the direction of the gradient ∇f_0 and decreases most rapidly in the opposite direction (see Figure 11.29).

EXAMPLE 4 Maximal rate of increase and decrease

In what direction is the function defined by $f(x, y) = xe^{2y-x}$ increasing most rapidly at the point $P_0(2, 1)$, and what is the maximum rate of increase? In what direction is f decreasing most rapidly?

Solution

We begin by finding the gradient of f:

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} = [e^{2y-x} + xe^{2y-x}(-1)]\mathbf{i} + [xe^{2y-x}(2)]\mathbf{j}$$
$$= e^{2y-x}[(1-x)\mathbf{i} + 2x\mathbf{j}]$$

At $(2, 1)$, $\nabla f_0 = e^{2(1)-2}[(1-2)\mathbf{i} + 2(2)\mathbf{j}] = -\mathbf{i} + 4\mathbf{j}$. The most rapid rate of increase is $\|\nabla f_0\| = \sqrt{(-1)^2 + (4)^2} = \sqrt{17}$ and it occurs in the direction of $-\mathbf{i} + 4\mathbf{j}$. The most rapid rate of decrease occurs in the direction of $-\nabla f_0 = \mathbf{i} - 4\mathbf{j}$.

FUNCTIONS OF THREE VARIABLES

The directional derivative and gradient concepts can easily be extended to functions of three or more variables. For a function of three variables, $f(x, y, z)$, the gradient ∇f is defined by

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$$

and the directional derivative $D_{\mathbf{u}}f$ of $f(x, y, z)$ at $P_0(x_0, y_0, z_0)$ in the direction of the unit vector $\mathbf{u}$ is given by

$$D_{\mathbf{u}}f = \nabla f_0 \cdot \mathbf{u}$$

where, as before, ∇f_0 is the gradient ∇f evaluated at P_0. The basic properties of the gradient of $f(x, y)$ (Theorem 11.8) are still valid, as is the maximal direction property of Theorem 11.9. Similar definitions and properties are valid for functions of more than three variables.

EXAMPLE 5 Directional derivative of a function of three variables

Let $f(x, y, z) = xy\sin(xz)$. Find ∇f_0 at the point $P_0(1, -2, \pi)$ and then compute the directional derivative of f at P_0 in the direction of the vector $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$.

Solution

Begin with the partial derivatives:

$$\begin{aligned} f_x &= y\sin(xz) + xy(z\cos(xz)); & f_x(1, -2, \pi) &= -2\sin\pi - 2\pi\cos\pi = 2\pi \\ f_y &= x\sin(xz); & f_y(1, -2, \pi) &= 1\sin\pi = 0 \\ f_z &= xy(x\cos(xz)); & f_z(1, -2, \pi) &= (1)(-2)(1)\cos\pi = 2 \end{aligned}$$

Thus, the gradient of f at P_0 is

$$\nabla f_0 = 2\pi\mathbf{i} + 2\mathbf{k}$$

To find $D_{\mathbf{u}}f$ we need $\mathbf{u}$, the unit vector in the direction of $\mathbf{v}$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{-2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}}{\sqrt{(-2)^2 + (3)^2 + (-5)^2}} = \frac{1}{\sqrt{38}}(-2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})$$

Finally,

$$D_{\mathbf{u}}f(1, -2, \pi) = \nabla f_0 \cdot \mathbf{u} = \frac{1}{\sqrt{38}}(-4\pi - 10) \approx -3.66$$

NORMAL PROPERTY OF THE GRADIENT

Suppose S is a level surface of the function defined by $f(x, y, z)$; that is, $f(x, y, z) = K$ for some constant K. Then if $P_0(x_0, y_0, z_0)$ is a point on S, the following theorem shows that the gradient ∇f_0 at P_0 is a vector that is **normal** (that is, orthogonal) to the tangent plane surface at P_0 (see Figure 11.30).

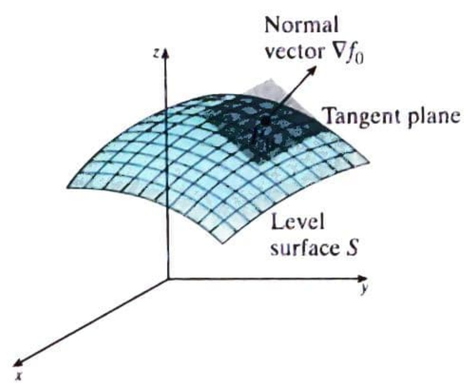


Figure 11.30 The normal property of the gradient

***THEOREM 11.10* The normal property of the gradient**

Suppose the function f is differentiable at the point P_0 and that the gradient at P_0 satisfies $\nabla f_0 \neq \mathbf{0}$. Then ∇f_0 is orthogonal to the level surface of f through P_0.

Proof Let C be any smooth curve on the level surface $f(x, y, z) = K$ that passes through $P_0(x_0, y_0, z_0)$, and describe the curve C by the vector function $\mathbf{R}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ for all t in some interval I. We will show that the gradient ∇f_0 is orthogonal to the tangent vector $d\mathbf{R}/dt$ at P_0.

Because C lies on the level surface, any point $P(x(t), y(t), z(t))$ on C must satisfy $f[x(t), y(t), z(t)] = K$, and by applying the chain rule, we obtain

$$\frac{d}{dt}[f(x(t), y(t), z(t))] = f_x(x, y, z)\frac{dx}{dt} + f_y(x, y, z)\frac{dy}{dt} + f_z(x, y, z)\frac{dz}{dt}$$

Suppose $t = t_0$ at P_0. Then

$$\frac{d}{dt}[f(x(t), y(t), z(t))]\bigg|_{t=t_0}$$
$$= f_x(x(t_0), y(t_0), z(t_0))\frac{dx}{dt} + f_y(x(t_0), y(t_0), z(t_0))\frac{dy}{dt} + f_z(x(t_0), y(t_0), z(t_0))\frac{dz}{dt}$$
$$= \nabla f_0 \cdot \frac{d\mathbf{R}}{dt}$$

since $\dfrac{d\mathbf{R}}{dt} = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j} + \dfrac{dz}{dt}\mathbf{k}$. We also know that $f(x(t), y(t), z(t)) = K$ for all t in I (because the curve C lies on the level surface $f(x, y, z) = K$). Thus, we have

$$\frac{d}{dt}\{f[x(t), y(t), z(t)]\} = \frac{d}{dt}(K) = 0$$

and it follows that $\nabla f_0 \cdot \dfrac{d\mathbf{R}}{dt} = 0$. We are given that $\nabla f_0 \neq \mathbf{0}$, and $d\mathbf{R}/dt \neq \mathbf{0}$ because the curve C is smooth. Therefore, ∇f_0 is orthogonal to $d\mathbf{R}/dt$, as required. ❑

> ➜ **What This Says** The gradient ∇f_0 at each point P_0 on the surface $f(x, y, z) = K$ is orthogonal at P_0 to the tangent vector $\mathbf{T} = \dfrac{d\mathbf{R}}{dt}$ of each curve C on the surface that passes through P_0. Thus, all these tangent vectors lie in a single plane through P_0 with normal vector $\mathbf{N} = \nabla f_0$. This plane is the *tangent plane* to the surface at P_0.

EXAMPLE 6 Finding a vector that is normal to a level surface

Find a vector that is normal to the level surface $x^2 + 2xy - yz + 3z^2 = 7$ at the point $P_0(1, 1, -1)$.

Solution

Since the gradient vector at P_0 is perpendicular to the level surface, we have

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k} = (2x + 2y)\mathbf{i} + (2x - z)\mathbf{j} + (6z - y)\mathbf{k}$$

At the point $(1, 1, -1)$, $\nabla f_0 = 4\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$ is the required normal. ■

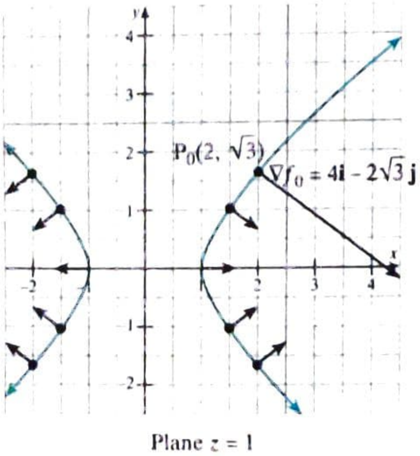


Figure 11.31 The level curve $x^2 - y^2 = 1$

Here is an example in which f involves only two variables, so $f(x, y) = K$ is a level curve in the plane instead of a level surface in space.

EXAMPLE 7 Finding a vector normal to a level curve

Sketch the level curve corresponding to $C = 1$ for the function $f(x, y) = x^2 - y^2$ and find a normal vector at the point $P_0(2, \sqrt{3})$.

Solution

The level curve for $C = 1$ is a hyperbola given by $x^2 - y^2 = 1$, as shown in Figure 11.31. The gradient vector is perpendicular to the level curve. We have

$$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} = 2x\mathbf{i} - 2y\mathbf{j}$$

so at the point $(2, \sqrt{3})$, $\nabla f_0 = 4\mathbf{i} - 2\sqrt{3}\mathbf{j}$ is the required normal. This normal vector and a few others are shown in Figure 11.31. ■

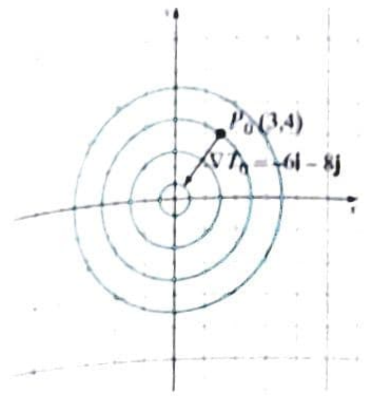


Figure 11.32 Isotherms of $T(x,y) = x^2 + y^2$. Heat flow at P_0 is in the direction of $-\nabla T_0 = -6\mathbf{i} - 8\mathbf{j}$.

EXAMPLE 8 Heat flow application

The set of points (x, y) with $0 \le x \le 5$ and $0 \le y \le 5$ is a square in the first quadrant of the xy-plane. Suppose this square is heated in such a way that $T(x, y) = x^2 + y^2$ is the temperature at the point $P(x, y)$. In what direction will heat flow from the point $P_0(3, 4)$?

Solution

The flow of heat in the region is given by a vector function $\mathbf{H}(x, y)$, whose value at each point (x, y) depends on x and y. From physics it is known that $\mathbf{H}(x, y)$ will be perpendicular to the isothermal curves $T(x, y) = C$ for C constant. The gradient ∇T and all its multiples point in such a direction. Therefore, we can express the heat flow as $\mathbf{H} = -k\nabla T$, where k is a positive constant (called the *thermal conductivity*). The negative sign is introduced to account for the fact that heat flows "downhill" (that is, in the direction of decreasing temperature).

Because $T(3, 4) = 25$, the point $P_0(3, 4)$ lies on the isotherm $T(x, y) = 25$, which is part of the circle $x^2 + y^2 = 25$, as shown in Figure 11.32. We know that the heat flow $\mathbf{H}_0$ at P_0 will satisfy $\mathbf{H}_0 = -k\nabla T_0$, where ∇T_0 is the gradient at P_0. Because $\nabla T = 2x\mathbf{i} + 2y\mathbf{j}$, we see that $\nabla T_0 = 6\mathbf{i} + 8\mathbf{j}$. Thus, the heat flow at P_0 satisfies

$$\mathbf{H}_0 = -k\nabla T_0 = -k(6\mathbf{i} + 8\mathbf{j})$$

Because the thermal conductivity k is positive, we can say that heat flows from P_0 in the direction of the unit vector $\mathbf{u}$ given by

$$\mathbf{u} = \frac{-(6\mathbf{i} + 8\mathbf{j})}{\sqrt{(-6)^2 + (-8)^2}} = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$ ■

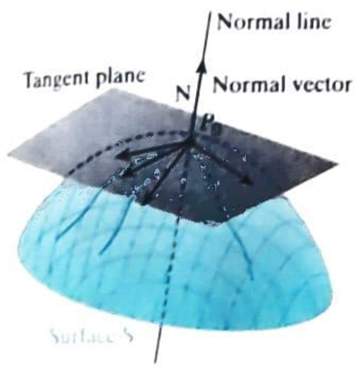


Figure 11.33 Tangent plane and normal line

TANGENT PLANES AND NORMAL LINES

Tangent planes and normal lines to a surface are the natural extensions to $\mathbb{R}^3$ of the tangent and normal lines we examined in $\mathbb{R}^2$. Suppose S is a surface and $\mathbf{N}$ is a vector normal to S at the point P_0. We would intuitively expect the normal line and the tangent plane to S at P_0 to be, respectively, the line through P_0 with the direction of $\mathbf{N}$ and the plane through P_0 with normal $\mathbf{N}$ (see Figure 11.33). These observations lead us to the following definition.

Tangent Plane and Normal Line to a Surface

> Suppose the surface S has a nonzero normal vector $\mathbf{N}$ at the point P_0. Then the line through P_0 parallel to $\mathbf{N}$ is called the **normal line** to S at P_0, and the plane through P_0 with normal vector $\mathbf{N}$ is the **tangent plane** to S at P_0.

We would expect a surface S with the representation $z = f(x, y)$ to have a nonvertical tangent plane at each point where $\nabla f \neq \mathbf{0}$. In particular, if S has an equation of the form $F(x, y, z) = C$, where C is a constant and F is a function differentiable at P_0, the normal property of a gradient tells us that the gradient ∇F_0 at P_0 is normal to S (if $\nabla F_0 \neq \mathbf{0}$) and that S must therefore have a tangent plane at P_0.

EXAMPLE 9 Finding the tangent plane and normal line to a given surface

Find equations for the tangent plane and the normal line at the point $P_0(1, -1, 2)$ on the surface S given by $x^2y + y^2z + z^2x = 5$.

Solution

We need to rewrite this problem so that the normal property of the gradient theorem applies. Let $F(x, y, z) = x^2y + y^2z + z^2x$, and consider S to be the level surface $F(x, y, z) = 5$. The gradient ∇F is normal to S at P_0. We find that

$$\nabla F(x, y, z) = (2xy + z^2)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2xz)\mathbf{k}$$

so a normal vector at P_0 is

$$\mathbf{N} = \nabla F_0 = \nabla F(1, -1, 2) = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

Hence, the required tangent plane is

$$2(x - 1) - 3(y + 1) + 5(z - 2) = 0 \qquad \text{or} \qquad 2x - 3y + 5z = 15$$

The normal line to the surface at P_0 is

$$x = 1 + 2t, \qquad y = -1 - 3t, \qquad z = 2 + 5t$$

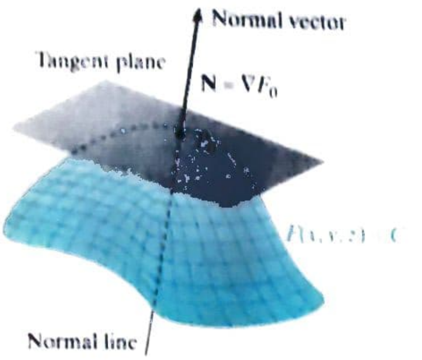


Figure 11.34 The tangent plane and normal line to a surface

By generalizing the procedure illustrated in the preceding example, we are led to the following formulas for the tangent plane and normal line. (Also see Figure 11.34.)

Formulas for the Tangent Plane and Normal Lines to a Surface

Suppose S is a surface with the equation $F(x, y, z) = C$ and let $P_0(x_0, y_0, z_0)$ be a point on S where F is differentiable with $\nabla F_0 \neq \mathbf{0}$. Then the **equation of the tangent plane** to S at P_0 is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

and the **normal line** to S at P_0 has parametric equations

$$\begin{aligned} x &= x_0 + F_x(x_0, y_0, z_0)t \\ y &= y_0 + F_y(x_0, y_0, z_0)t \\ z &= z_0 + F_z(x_0, y_0, z_0)t \end{aligned}$$

Note if $z = f(x, y)$, we have $F(x, y, z) = f(x, y) - z = 0$. Then $F_x = f_x$, $F_y = f_y$, and $F_z = -1$ and the equation of the tangent plane becomes

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) - (z - z_0) = 0$$

which is equivalent to the tangent plane formula given in Section 11.4.

EXAMPLE 10 Equations of the tangent plane and the normal line

Find the equations for the tangent plane and the normal line to the cone $z^2 = x^2 + y^2$ at the point where $x = 3$, $y = 4$, and $z > 0$.

Solution

If $P_0(x_0, y_0, z_0)$ is the point of tangency and $x_0 = 3$, $y_0 = 4$, and $z_0 > 0$, then

$$z_0 = \sqrt{x_0^2 + y_0^2} = \sqrt{9 + 16} = 5$$

If we consider $F(x, y, z) = x^2 + y^2 - z^2$, then the cone can be regarded as the level surface $F(x, y, z) = 0$. The partial derivatives of F are

$$F_x = 2x \qquad F_y = 2y \qquad F_z = -2z$$

so at $P_0(3, 4, 5)$,

$$F_x(3, 4, 5) = 6, \qquad F_y(3, 4, 5) = 8, \qquad F_z(3, 4, 5) = -10$$

Thus the tangent plane has the equation

$$6(x - 3) + 8(y - 4) - 10(z - 5) = 0$$

or $3x + 4y - 5z = 0$, and the normal line is given parametrically by the equations

$$x = 3 + 6t, \qquad y = 4 + 8t, \qquad z = 5 - 10t$$