The Completeness property of $\mathbb{R}$

Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.

Ex: Let $S \neq \phi \subseteq \mathbb{R}$, S be bounded above and let $a \in \mathbb{R}$ and

$$a + S = \{a + x : x \in S\}$$

prove that $\sup(a+S) = a + \sup S$.

Pf: Let $u = \sup S$.

Then $x \leq u$ for all $x \in S$.

$\therefore$ $a + x \leq a + u$ for all $x \in S$

$\Rightarrow$ $a + u$ is an upper bound of $a + S$.

$\therefore$ $\sup(a+S) \leq a + u = a + \sup S$ — (I)

Now, if v is any upper bound of the set $a + S$, then $a + x \leq v$ for all $x \in S$.

$\Rightarrow x \leq v - a$ for all $x \in S$

$\therefore$ $v - a$ is an upper bound of S.

$\Rightarrow \sup S \leq v - a$

$\Rightarrow \cancel{a + \sup S \leq v}$ — (II)

$\Rightarrow u = \sup S \leq v - a$

$\Rightarrow a + u \leq v.$

since v is any upper bound of $a+S$, so we can replace v by $\sup(a+S)$ to get

$$a + u \leq \sup(a+S)$$

$$\Rightarrow a + \sup S \leq \sup(a+S) \quad \text{---(II)}$$

from (I) & (II) we get

$$\sup(a+S) = a + \sup S$$

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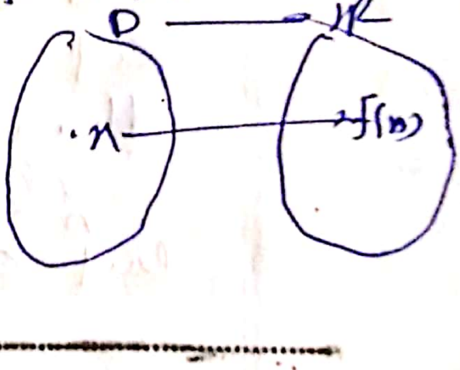
Bounded above and bounded below function

$$f : D \to \mathbb{R}$$

we say that f is bounded above if the set $f(D) = \{f(x) : x \in D\}$ is bounded above in $\mathbb{R}$ i.e. if there exists $M \in \mathbb{R}$ s.t. $f(x) \leq M$ for all $x \in D$. Similarly the function f is bounded below if the set $f(D) = \{f(x) : x \in D\}$ is bounded below in $\mathbb{R}$.



Teacher's Signature..................

Archimedian property:

If $x \in \mathbb{R}$, Then there exists $n_x \in \mathbb{N}$ such That $x < n_x$.

Pf:- If the assertion is false, then $n \leq x$ for all $n \in \mathbb{N}$. Therefore x is an upper bound of $\mathbb{N}$, the set of natural number. {By the completeness property} $\mathbb{N}$ has a supremum $u \in \mathbb{R}$. Since $u - 1 < u$, so $u - 1$ is not an upper bound of $\mathbb{N}$, so There exists $m \in \mathbb{N}$ s.t. $u - 1 < m \Rightarrow u < m + 1$.

But $m + 1 \in \mathbb{N}$, so $u < m + 1$ Contradicts the fact that u is ~~the supremum~~ an upper bound of $\mathbb{N}$. Hence there must exist $n_x \in \mathbb{N}$ s.t.

$$x < n_x.$$

Corollary: If $S = \{ \frac{1}{n} : n \in \mathbb{N} \}$, then $\inf S = 0$.

Pf: Since $S \neq \phi$ is bounded below by 0, so it has an infimum and let $\omega = \inf S$. Clearly, $\omega \geq 0$. and $\omega \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then $\frac{1}{\varepsilon} \in \mathbb{R}$ By Archimedian property there exists $n \in \mathbb{N}$ s.t. $\frac{1}{\varepsilon} < n$; which implies $\frac{1}{n} < \varepsilon$. Therefore we have $0 \leq \omega \leq \frac{1}{n} < \varepsilon$. But since $\varepsilon > 0$ is arbitrary, so $\omega = 0 \Rightarrow \inf S = 0$.