Thm: Let $n$ and $k$ be fixed positive integer with $k < n$. Let $m$ be a number between $0$ and $\binom{n}{k}-1$. Then there exist unique integer $\alpha_i$, $i = 1, 2, \ldots k$ such that

$$n-1 \geq \alpha_1 > \alpha_2 > \ldots \ldots > \alpha_k \geq 0.$$

and satisfying

$$m = \binom{\alpha_1}{k} + \binom{\alpha_2}{k-1} + \ldots + \binom{\alpha_k}{1}.$$

Pf: Given $n$ and $k$ are fixed positive integer. Now fixed a $k$-subset $\{a_1, a_2, \ldots, a_k\}$ of $[n]$. Our aim is to find $k$-subset $\{b_1, b_2, \ldots, b_k\}$ succeed the given $k$-subset $\{a_1, a_2, \ldots a_k\}$ in lex order.

Now if $b_1 > a_1$ i.e. $\{b_1, b_2, \ldots, b_k\}$ is a subset of $\{a_1+1, a_1+2, \ldots, n\}$. Hence in this case no of such $k$-subset equal to $\binom{n-a_1}{k}$

If $b_1 = a_1$ and $b_2 > a_2$, then $\{b_2, b_3, \ldots, b_k\}$ is a subset of $\{a_2+1, a_2+2, \ldots, n\}$. Hence the no. of such $k$-subset equal to $\binom{n-a_2}{k}$.

In general, for some (fixed) $j$ between $1$ and $k$; $b_i = a_i$ for all $i$ from $1$ to $j-1$, and $b_j > a_j$. Then $\{b_j, \ldots, b_k\}$ is a subset of $\{a_j+1, a_j+2, \ldots n\}$. Therefore the no of such $k$-subset is equal to $\binom{n-a_j}{k-j+1}$. where $1 \leq j \leq k$

Thus total no of subset succeeding the given set is $\binom{n-a_1}{k} + \binom{n-a_2}{k-1} + \ldots + \binom{n-a_j}{n-j+1} + \ldots \binom{n-a_k}{1}$

Now for any $k$-subset $\{a_1, a_2, \ldots a_k\}$ of $[n]$, the no of $k$-subset that succeed it in lex order is uniquely determined by that $k$-subset. ~~Since~~ the no. of $k$-subset succeeding a given $k$ subset can take ~~to~~ values from $0$ to $\binom{n}{k}-1$ (depending upon the given $k$-subset).

Now consider a no. $m$ such that $0 \leq m \leq \binom{n}{k}-1$.

Let $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ denote the sequence obtained from $\{a_1, a_2, \dots a_k\}$ by letting $\alpha_i = n - a_i$

Let $A$ = set of k-subset succeeding a given k-subset.

$B$ = set of no. from 0 to $\binom{n}{k}-1$.

Since from the above discussion we observed that there is a 1-1 correspondence between the set A and set B.

Thus for any no m $(0 \le m \le \binom{n}{k}-1)$

$$m = \binom{\alpha_1}{k} + \binom{\alpha_2}{k-1} + \dots + \binom{\alpha_k}{1}.$$

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Thm: Let n and k be fixed positive integers with $k<n$. Then for any number m between 0 and $\binom{n}{k}-1$, there exist unique integers $\beta_i$, $i=1,2,\dots k$ such that

$$0 \le \beta_1 < \beta_2 < \dots < \beta_k \le n-1$$

and satisfying

$$m = \binom{\beta_k}{k} + \binom{\beta_{k-1}}{k-1} + \dots + \binom{\beta_1}{1}$$

Pf: Given n and k are fixed numbers.

Now fixed a k-subset $\{a_1, a_2, \dots, a_k\}$ of $[n]$.

Our aim is to find the k-subset $\{b_1, b_2, \dots, b_k\}$ preceed the given k-subset $\{a_1, a_2, \dots, a_k\}$ in colex order.

If $b_k < a_k$, then $\{b_1, \dots b_k\} \subseteq \{1, \dots a_k-2, a_k-1\}$.

Hence in this no of such k-subset is equal to $\cancel{\binom{a_k-1}{k}}$. $\binom{a_k-1}{k}$

~~If not, i.e. $c_k = d_k$,~~

If not, $a_k = b_k$ and $b_{k-1} < a_{k-1}$, then $\{b_1, b_2, \dots b_{k-1}\}$ $\subseteq \{1, 2, \dots\ a_{k-1}-2, a_{k-1}-1\}$. Hence no of such k-subset is equal to $\binom{a_{k-1}-1}{k-1}$.

In general, for some fixed $j$, between $1 \le j \le k$;
$a_k = b_k$, $a_{k-1} = b_{k-1}$, $\dots\ a_{j-1} = b_{j-1}$ and $b_j < a_j$,
then $\{b_1, b_2 \dots, b_j\} \subseteq \{1, 2, \dots\ a_j - 1\}$.
Hence the no of such k-subset is equal to

$$\binom{a_j-1}{j}.$$

Hence the total no of k-subset preceding $\{a_1, a_2, \dots a_k\}$ is equal to

$$\sum_{j=1}^{j=k} \binom{a_j-1}{j}$$

~~Thus the total no of subset~~

Thus for any k-subset $\{a_1, a_2, \dots, a_k\}$ of $[n]$, the total no. of k-subset that succeed it in colex order is uniquly determined by that k-subset.
Thus the no. of k-subset preceding the set $\{a_1, a_2, \dots, a_k\}$ can take values from $0$ to $\binom{n}{k}-1$.
Now consider a no. of $m$ such that $0 \le m \le \binom{n}{k} - 1$.

Let $\{\beta_1, \beta_2, \dots \beta_k\}$ denote the sequence obtained from $\{a_1, a_2, \dots a_k\}$ by letting $\beta_j = a_j - 1$

Let $A$ = set of k-subset succeding a given k-subset

$B$ = set of no. from $0$ to $\binom{n}{k} - 1$.

Since from the above discussion, we observed that there is 1-1 corresponds beth set A & B. Thus for any $m$ $\left(0 \le m \le \binom{n}{k}-1\right)$

$$m = \binom{\beta_1}{k} + \binom{\beta_2}{k-1} + \dots + \binom{\beta_!}{1}.$$