Integrating Factors:

Given the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

if $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$

then the equation is exact and we can obtain a one-parameter family of solutions.

But if $\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$

then the equation is not exact. What shall we do in such a case? Perhaps we can multiply the non-exact equation by some expression that will transform it into an essentially equivalent exact equation. If so, we can proceed to solve the resulting exact equation by the above procedures.

Let us consider the equation

$$y\,dx + 2x\,dy = 0$$

this equation is not exact. However, if we multiply this equation by y, it is transformed into the essentially equivalent equation

$$y^2\,dx + 2xy\,dy = 0.$$

which is exact. Since this resulting exact equation is integrable, we call y an integrating factor of this equation.

Def^n: If the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$ is not exact

in a domain D but the differential equation

$$\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$$

is exact in D, then $\mu(x,y)$ is called an integrating factor of the above differential equation, Consider the differential equation

$$(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0$$

This eq^n is of the form $M(x,y)\,dx + N(x,y)\,dy = 0$, where

$M(x,y) = 3y + 4xy^2$ and $N(x,y) = 2x + 3x^2y$

Here, $\dfrac{\partial M(x,y)}{\partial y} = 3 + 8xy$, $\dfrac{\partial N(x,y)}{\partial x} = 2 + 6xy$

Since $\dfrac{\partial M(x,y)}{\partial y} \neq \dfrac{\partial N(x,y)}{\partial x}$ except for (x,y) such that $2xy + 1 = 0$, ∴ above equation is not exact in any rectangular domain D.

Let $\mu(x,y) = x^2y$. Then the corresponding differential equation is of the form

$$(3x^2y^2 + 4x^3y^3)\,dx + (2x^3y + 3x^4y^2)\,dy = 0$$

This equation is exact in every rectangular domain D, since

$$\frac{\partial[\mu(x,y)M(x,y)]}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial[\mu(x,y)N(x,y)]}{\partial x}$$

for all real (x,y). Hence $\mu(x,y)$ is an integrating factor.

Multiplication of a non-exact differential equation by an integrating factor thus transforms the non-exact equation into an exact one. We have referred to this resulting exact equation as essentially equivalent to the original.