Remarks: Two following two statements about a number u and a set S are equivalent:

(I) u is an upper bound of S.

(I)' $x \leq u$ for all $x \in S$.

The following statements about an upper bound u of a set S are equivalent:

(I) if v is any upper bound of S, then $u \leq v$

(II) if $z < u$, then z is not an upper bound of S

(III) if $z < u$, then there exists $x_z \in S$ s.t. $z < x_z$.

(IV) if $\varepsilon > 0$, then there exists $x_\varepsilon \in S$ s.t. $u + \varepsilon < x_\varepsilon$

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Lemma: A number u is the supremum of a nonempty subset S of $\mathbb{R}$ if and only if u satisfies the conditions:

(1) $x \leq u$ for all $x \in S$

(2) if $v < u$, then there exists $x' \in S$ s.t. $v < x'$.

Pf:

$\Rightarrow$ Let u be the supremum of S.

$\Rightarrow$ u is the least upper bound of S.

So by definition of upper bound

$\rightarrow$ $x \leq u$ for all $x \in S$. [(1) is proved]

Next, let $v < u$. we need to show there exists $x' \in S$ s.t. $v < x'$.

If possible let there doesnot exist any element $x' \in S$ s.t. $v < x'$.

Then $v \geq x^*$ for all $x^* \in S$.

$\Rightarrow x \leq v$, for all $x \in S$

$\Rightarrow$ v is an upper bound of S,

which is a contradiction as $v < u$ and u is the least upper bound of S

Hence if $v < u$, then always exists $x' \in S$ s.t. $v < x'$.

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Conversely let statements ① & ② are ~~[illegible]~~ satisfied. To show u is the supremum of S.

From statement ① we get

$$x \leq u \text{ for all } x \in S$$

$\Rightarrow u$ is an upper bound of S.

From statement ② we get

if $v < u$, then there exists $x' \in S$ s.t. $v < x'$

$\Rightarrow v$ is not an upper bound of S.

i.e. no number less than u can be an upper bound of S.

$\Rightarrow u$ is the least upper bound of S

$\Rightarrow u$ is the supremum of S.

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