Theorem: Let a, b and c belong to an integral domain R. If $a \neq 0$ and $ab = ac$, then $b = c$.

Pf: Let $ab = ac$, $a \neq 0$

$$a(b-c) = ab - ac$$
$$= 0 \quad [\because ab = ac]$$

Since R is an integral domain and $a \neq 0$

so $a(b-c) = 0 \Rightarrow b - c = 0$

$$\Rightarrow b = c \quad \#$$

Ex: Show that a commutative ring with the cancellation property (under multiplication) has no zero divisors.

Sol: Let R be a commutative ring with the cancellation property (under multiplication)
i.e. for $a, b, c \in R$ $ab = ac \Rightarrow b = c$.
To show R has no zero divisors.
Let $ab = 0$ and $a \neq 0$.
Now, $ab = 0 = a \cdot 0$
$\Rightarrow b = 0$ [by Cancellation prop]
$\therefore$ R has no zero divisors $\#$

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Definition: An element a of a ring R with unity 1 is said to be unit (or invertible) if there exists an element b of R such that $ab = ba = 1$.

eg. Consider the ring $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ under addition and multiplication modulo 8.

$\times_8$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

From the table we have seen that

$1 \times_8 1 = 1$, $3 \times_8 3 = 1$, $5 \times_8 5 = 1$, $7 \times_8 7 = 1$

$\therefore$ 1, 3, 5, 7 are units.

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Field. A commutative ring with a unity is called a field if every nonzero element is unit (i.e. if every nonzero element is invertible).

Theorem: Every field is an integral domain.

Pf. Let R be a field. So R is commutative ring with unity and every nonzero element of R is invertible. To show R is an integral domain, we need to show R has no zero divisors.

for $a, b \in R$, let $ab = 0$.

If $a \neq 0$, then since R is a field so a^{-1} exists.

$$\therefore \quad ab = 0$$
$$\Rightarrow a^{-1}(ab) = a^{-1}0$$
$$\Rightarrow (a^{-1}a)\, b = 0$$
$$\Rightarrow 1\, b = 0$$
$$\Rightarrow b = 0$$

Hence R has no zero divisors.

$\Rightarrow$ R is an integral domain. #

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Examples of field

(1) Set of Complex numbers is a field under usual addition and multiplication.

(2) Set of Real numbers, set of rational numbers are fields under usual addition and multiplication.

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