

Ex) Prove the following identities:

$$(i) \quad u_0 + \frac{u_1 x}{1} + \frac{u_2 x^2}{2} + \frac{u_3 x^3}{3} + \dots = e^x \left[u_0 + x \Delta u_0 + \frac{x^2}{2} \Delta^2 u_0 + \frac{x^3}{3} \Delta^3 u_0 + \dots \right]$$

$$(ii) \quad u_1 x + u_2 x^2 + u_3 x^3 + \dots = \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2} \Delta u_1 + \frac{x^3}{(1-x)^3} \Delta^2 u_1 + \dots$$

for $0 < x < 1$

Sol^m:

(i) we have,

L.H.S.

$$\begin{aligned}
 & u_0 + \frac{u_1 x}{L} + \frac{u_2 x^2}{L^2} + \frac{u_3 x^3}{L^3} + \dots \\
 &= u_0 + \frac{E u_0 x}{L} + \frac{E^2 u_0 x^2}{L^2} + \frac{E^3 u_0 x^3}{L^3} + \dots \\
 &= u_0 \left[1 + \frac{x E}{L} + \frac{x^2 E^2}{L^2} + \frac{x^3 E^3}{L^3} + \dots \right] \\
 &= e^{x E} u_0 = \left\{ e^{x(1+\Delta)} \right\} u_0 = e^x e^{x \Delta} u_0 \\
 &= e^x \left[1 + x \Delta + \frac{x^2 \Delta^2}{2} + \frac{x^3 \Delta^3}{6} + \dots \right] u_0 \\
 &= e^x \left[u_0 + x \Delta u_0 + \frac{x^2 \Delta^2 u_0}{2} + \frac{x^3 \Delta^3 u_0}{6} + \dots \right] = R.H.S.
 \end{aligned}$$

(ii) Sol^m:-

we have,

R.H.S.

$$\begin{aligned}
 & \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2} (E-1) u_1 + \frac{x^3}{(1-x)^3} (E-1)^2 u_1 + \dots \\
 &= \left\{ \frac{x}{1-x} + \frac{x^2}{(1-x)^2} + \frac{x^3}{(1-x)^3} + \dots \right\} u_1 \\
 &+ \left\{ \frac{x^2}{(1-x)^2} - \frac{2x^3}{(1-x)^3} + \dots \right\} E u_1 + \left\{ \frac{x^3}{(1-x)^3} - \dots \right\} E^2 u_1 + \dots \\
 &= \frac{x}{1-x} \left(1 + \frac{x}{1-x} \right)^{-1} u_1 + \frac{x^2}{(1-x)^2} \left(1 + \frac{x}{1-x} \right)^{-2} u_2 \\
 &+ \frac{x^3}{(1-x)^3} \left(1 + \frac{x}{1-x} \right)^{-3} u_3 + \dots \\
 &= \frac{x}{1-x} \cdot \frac{1-x}{1} u_1 + \frac{x^2}{(1-x)^2} \frac{(1-x)^2}{1} u_2 + \dots \\
 &= u_1 x + u_2 x^2 + u_3 x^3 + \dots = L.H.S. =
 \end{aligned}$$

Ex) with usual notations prove that

$$\begin{aligned}
 & u_0 + u_1 x + u_2 x^2 + \dots \rightarrow \infty \\
 &= \frac{u_0}{1-x} + \frac{x \Delta u_0}{(1-x)^2} + \frac{x^2 \Delta^2 u_0}{(1-x)^3} + \dots \rightarrow \infty
 \end{aligned}$$

Solⁿ: L.H.S. =

$$u_0 + u_1 x + u_2 x^2 + \dots \text{ to } \infty$$

$$= u_0 + x E u_0 + x^2 E^2 u_0 + \dots \text{ to } \infty$$

$$= [1 + x E + (x E)^2 + \dots \text{ to } \infty] u_0$$

$$= [(1 - x E)^{-1}] u_0$$

$$= \frac{1}{1 - x E} u_0 = \frac{1}{1 - x(1 + \Delta)} u_0$$

$$= \frac{1}{1 - x - x\Delta} u_0 = \frac{1}{1 - x} \left[1 - \frac{x\Delta}{1 - x} \right]^{-1} u_0$$

$$= \frac{1}{1 - x} \left[1 + \frac{x\Delta}{1 - x} + \frac{x^2 \Delta^2}{(1 - x)^2} + \dots \text{ to } \infty \right] u_0$$

$$= \frac{u_0}{1 - x} + \frac{x\Delta u_0}{(1 - x)^2} + \frac{x^2 \Delta^2 u_0}{(1 - x)^3} + \dots \text{ to } \infty$$

R.H.S. = $[1 + \Delta + \Delta^2 + \dots \text{ to } \infty] u_0$

Ex) Establish the relation $E \equiv e^{hD}$ where D is the differential operator, Hence show that

$$D = \frac{1}{h} \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots \right)$$

S.I^m: Here,

$$Df(x) = \frac{d}{dx} f(x) \text{ i.e.}$$

$D \equiv \frac{d}{dx}$ is called the differential operator.

Now by Taylor's theorem of differential calculus, we have,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \dots$$

$$= \left[f(x) + h Df(x) + \frac{h^2}{2} D^2 f(x) + \dots \right]$$

$$= \left(1 + hD + \frac{h^2}{2} + \dots \right) f(x)$$

$$\Rightarrow f(x+h) = e^{hD} f(x) \quad \dots (1)$$

In calculus of finite differences,

$$f(x+h) = E f(x) \quad \dots (2)$$

Comparing (1) and (2), we get,

$$E f(x) = e^{hD} f(x)$$

$$\text{i.e. } E \equiv e^{hD}$$

Again, we have,

$$E f(x) = f(x+h) \quad \dots (A)$$

and $\Delta f(x) = f(x+h) - f(x)$

$$\Rightarrow \Delta f(x) + f(x) = f(x+h) \quad \dots (B)$$

Comparing (A) and (B), we get,

$$E f(x) = (1 + \Delta) f(x)$$

$$\text{i.e. } E \equiv 1 + \Delta$$

$$hD = \log(1 + \Delta) = \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots \right)$$

$$\Rightarrow D = \frac{1}{h} \left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots \right]$$

Ex) Find the function $f(x)$ whose first difference $\Delta f(x)$ is

$$9x^2 + 11x + 5$$

Sol: Here $\Delta f(x) = 9x^2 + 11x + 5 \quad \dots (1)$

We know that if degree of $f(x) = n$ then degree of $[\Delta f(x)] = n-1$.

$$\text{Here, } \deg[\Delta f(x)] = 2 \quad \therefore \deg f(x) = 3$$

$$\text{Let us put } f(x) = ax^3 + bx^2 + cx + d \quad \dots (2)$$

where a, b, c, d are constants and $a \neq 0$.

Let the interval of differencing be 1 ($=h$), then

$$\Delta f(x) = f(x+1) - f(x)$$

$$= a(x+1)^3 + b(x+1)^2 + c(x+1) + d - ax^3 - bx^2 - cx - d$$

$$\Rightarrow \Delta f(x) = 3ax^2 + (3a+2b)x + (a+b+c) \quad \dots (3)$$

Comparing (1) and (3), we get

$$3a = 9, \quad 3a + 2b = 11 \quad \text{and} \quad a + b + c = 5$$

$$\Rightarrow a = 3$$

$$\Rightarrow b = 1$$

$$\Rightarrow c = 1$$

$\therefore f(x) = 3x^3 + x^2 + x + d$, which is the reqd. function.