Ex) Solve the initial-value problem:

$$(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy;$$

$$y(0) = 2$$

Sol^n: Here, $M(x,y) = 2x\cos y + 3x^2y$

$N(x,y) = x^3 - x^2\sin y - y$

Now, $\dfrac{\partial M(x,y)}{\partial y} = -2x\sin y + 3x^2$

$\dfrac{\partial N(x,y)}{\partial x} = 3x^2 - 2x\sin y$

we observe that the equation is exact in every rectangular domain D, since $\dfrac{\partial M(x,y)}{\partial y} = \dfrac{\partial N(x,y)}{\partial x}$ for all real (x,y).

Now, we must find F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 2x\cos y + 3x^2y$$

and $\dfrac{\partial F(x,y)}{\partial y} = N(x,y) = x^3 - x^2\sin y - y$

Then,
$$F(x,y) = \int M(x,y)\,\partial x + \phi(y)$$

$$= \int (2x\cos y + 3x^2y)\,\partial x + \phi(y)$$

$$= x^2\cos y + x^3y + \phi(y)$$

$\therefore$ $\dfrac{\partial F(x,y)}{\partial y} = -x^2\sin y + x^3 + \dfrac{d\phi(y)}{dy}$

But also, $\dfrac{\partial F(x,y)}{\partial y} = N(x,y) = x^3 - x^2\sin y - y$

and so, $x^3 - x^2\sin y - y = -x^2\sin y + x^3 + \dfrac{d\phi(y)}{dy}$

$$\Rightarrow \frac{d\phi(y)}{dy} = -y$$

$$\therefore \phi(y) = -\frac{y^2}{2} + c_0$$

$$\therefore F(x,y) = x^2\cos y + x^3y - \frac{y^2}{2} + c_0$$

Hence a one-parameter family of solution is

$F(x, y) = c_1$, which may be expressed as

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = c \quad , \text{ when } c = c_1 - c_0$$

Given that $y = 2$ when $x = 0$

$$\therefore \quad c = -\frac{4}{2} = -2$$

Thus the solution of the given initial-value problem is

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = -2$$ //

Exercise:

Determine whether or not each of the given eq's is exact; solve those that are exact.

1. $(3x + 2y)\,dx + (2x + y)\,dy = 0$
2. $(y^2 + 3)\,dx + (2xy - 4)\,dy = 0$
3. $(2xy + 1)\,dx + (x^2 + 4y)\,dy = 0$
4. $(6xy + 2y^2 - 5)\,dx + (3x^2 + 4xy - 6)\,dy = 0$

Exercise:

Solve the initial-value problems:

1. $(2xy - 3)\,dx + (x^2 + 4y)\,dy = 0$, $\quad y(1) = 2$
2. $(3x^2y^2 - y^3 + 2x)\,dx + (2x^3y - 3xy^2 + 1)\,dy = 0$, $\quad y(-2) = 1$
3. $(ye^x + 2e^x + y^2)\,dx + (e^x + 2xy)\,dy = 0$, $\quad y(0) = 6$.