Algebraic and Transcendental eq^n s:

An expression of the form $f_n(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$, where a's are constants, and n is a positive integer is said to be a polynomial in x of degree n provided $a_0 \neq 0$. The values of x which make $f_n(x)$ to zero are called as zeroes or the roots of the polynomial $f_n(x)$ and every polynomial of nth degree has n zeroes.

The eq^n s of the form $f_n(x) = 0$ are called Algebraic or Transcendental according as $f_n(x)$ is purely a polynomial in x or contains some other f^n s such as logarithmic, exponential and trigonometric f^n s etc. For example, the eq^n s $x^3 + 3x^2 + 2x + 4 = 0$ and $6x^3 + \log(x+3) + e^{-2x} + \cos x = 0$ are called algebraic and transcendal respectively.

Method of Bisection:

This method is based on the the^m which states that " If a f^n $f(x)$ is continuous in the closed interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs [i.e. if $f(a) f(b) < 0$] then there exists at least one real root of $f(x) = 0$ bet^n a and b."

Let $f(a)$ and $f(b)$ be of opposite signs, i.e. $f(a) f(b) < 0$.

Let $\xi_1 = \dfrac{a+b}{2}$, be the middle point of $[a, b]$.

Now, if $f(\xi_1) = 0$, then ξ_1 is a root of $f(x) = 0$. If $f(\xi_1) \neq 0$, then either $f(a) f(\xi_1) < 0$ or $f(\xi_1) f(b) < 0$.

If $f(a) f(\xi_1) < 0$, then the root will lie betn (a, ξ_1) and if $f(b) f(\xi_1) < 0$, then the root lies betn (ξ_1, b). We thus reduce the interval from $[a, b]$ to $[a, \xi_1]$ or $[\xi_1, b]$.

Then, as before, the newly reduced interval in which the root lies is again halved and the process is repeated until the root is obtained to the desired accuracy.

Ex) Solve the eqn $x^3 - 9x + 1 = 0$ for the root lying betn 2 and 3, correct to three significant figures.

Soln: We have,

$$f(x) = x^3 - 9x + 1 \quad \text{------ (1)}$$

$$f(2) = -9, \quad f(3) = 1$$

$$\therefore \quad f(2) f(3) < 0$$

A root of the eqn (1) lies betn 2 and 3.

Let $a_0 = 2$, $b_0 = 3$

Now,

$$m_1 = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(2+3) = 2.5$$

$$f(m_1) = f(2.5) = (2.5)^3 - 9(2.5) + 1$$

$$= -5.875$$

$$\therefore \quad f(2.5) f(3) < 0$$

Thus, the root lies in the interval $(2.5, 3)$.

Taking $a_1 = 2.5$, $b_1 = 3$, we get,

$$m_2 = \frac{1}{2}(2.5 + 3) = 2.75$$

$$f(2.75) = (2.75)^3 - 9(2.75) + 1$$

$$= -2.953$$

and $f(2.75) f(3) < 0$

Thus the root lies in the interval $(2.75, 3)$.

The sequence of the intervals is given below:

n	a_n	b_n	$m_{n+1} = \left(\frac{a_n+b_n}{2}\right)$	$f(m_{n+1})$
0	2	3	2.5	−5.8
1	2.5	3	2.75	−2.9
2	2.75	3	2.88	−1.03
3	2.88	3	2.94	−0.05
4	2.94	3	2.97	.47
5	2.94	2.97	2.955	.21
6	2.94	2.955	2.9475	.08
7	2.94	2.9475	2.9438	.017
8	2.94	2.9438	2.9419	.016

In the 8th step a_n, b_n and m_{n+1} are equal upto three significant figures. We can take 2.94 as a root upto three significant figures.

∴ The root of $x^3 - 9x + 1 = 0$ is 2.94 //