Ex) Prove that

$$\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$$

Sol^n:- R.H.S. $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

$$= \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}$$

$$= \frac{E-1}{\frac{E-1}{E}} - \frac{\frac{E-1}{E}}{E-1}$$

$$= E - E^{-1}$$

$$= (E-1) + (1-E^{-1})$$

$$= \Delta + \nabla \quad = \text{L.H.S.}$$

Ex) Evaluate ; $\Delta^2 E e^x \cdot \frac{5!\, e^x}{\Delta^2 e^x}$

Sol^n: Suppose 'h' is the interval of differencing.

Now, $\Delta e^x = e^{x+h} - e^x$

$$\therefore \Delta^2 e^x = \Delta(\Delta e^x) = \Delta(e^{x+h} - e^x)$$

$$= \Delta(e^{x+h}) - \Delta(e^x)$$

$$= e^{x+2h} - e^{x+h} - e^{x+h} + e^x$$

$$= e^{x+2h} - 2e^{x+h} + e^x$$

$$= e^x(e^{2h} - 2e^h + 1) \quad \cdots (1)$$

Again, $E e^x = e^{x+h}$

$$\therefore \Delta E e^x = \Delta(e^{x+h}) = e^{x+2h} - e^{x+h}$$

$$\therefore \Delta^2 E e^x = \Delta(e^{x+2h} - e^{x+h}) = e^{x+3h} - e^{x+2h} - e^{x+2h} + e^{x+h}$$

$$= e^{x+h}(e^{2h} - 2e^h + 1) \quad \cdots (2)$$

$$\therefore \Delta^2 E e^x \cdot \frac{5!\, e^x}{\Delta^2 e^x}$$

$$= e^{x+h}(e^{2h}-2e^h+1) \cdot \frac{5!\, e^x}{e^x(e^{2h}-2e^h+1)}$$

$$= 5!\, e^{x+h}$$

//

Note:- For convenience we may represent the function $f(x)$ by u_x so that $f(x+mh)$ is represented by u_{x+mh} and $f(0)$, $f(1)$, $f(2)$, etc may be represented by u_0, u_1, u_2 etc.

Ex) Prove with usual notations that

$$u_0 - u_1 + u_2 - \cdots = \frac{1}{2}u_0 - \frac{1}{4}\Delta u_0 + \frac{1}{8}\Delta^2 u_0 - \cdots$$

Solⁿ:- L.H.S. =

$$u_0 - u_1 + u_2 - \cdots$$

$$= u_0 - Eu_0 + E^2 u_0 - E^3 u_0 + \cdots$$

$$= (1 - E + E^2 - E^3 + \cdots)u_0$$

$$= (1+E)^{-1} u_0$$

$$= (2+\Delta)^{-1} u_0$$

$$= \frac{1}{2}\left(1+\frac{\Delta}{2}\right)^{-1} u_0$$

$$= \frac{1}{2}\left[1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \frac{\Delta^3}{8} + \cdots\right]u_0$$

$$= \left(\frac{1}{2} - \frac{\Delta}{4} + \frac{\Delta^2}{8} - \frac{\Delta^3}{16} + \cdots\right)u_0$$

$$= \frac{1}{2}u_0 - \frac{1}{4}\Delta u_0 + \frac{1}{8}\Delta^2 u_0 - \cdots = \text{R.H.S.}$$

//

Ex) Prove that,

$$\sum_{x=0}^{\infty} u_{2x} = \frac{1}{2}\sum_{x=0}^{\infty} u_x + \frac{1}{4}\left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \cdots\right)u_0$$

Solⁿ:- R.H.S. =

$$\frac{1}{2}(u_0 + u_1 + u_2 + u_3 + \cdots) + \frac{1}{4}\left(1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \cdots\right)u_0$$

$$= \frac{1}{2}\left\{u_0 + Eu_0 + E^2 u_0 + E^3 u_0 + \cdots\right\} + \frac{1}{4}\left(1+\frac{\Delta}{2}\right)^{-1} u_0$$

$$= \frac{1}{2}\left[1+E+E^2+E^3+\cdots\right]u_0 + \frac{1}{4}\left(1+\frac{\Delta}{2}\right)^{-1}u_0$$

$$= \frac{1}{2}\left[1-E\right]^{-1}u_0 + \frac{1}{4}\left(1+\frac{\Delta}{2}\right)^{-1}u_0$$

$$= \frac{1}{2}(1-E)^{-1}u_0 + \frac{1}{2}(2+\Delta)^{-1}u_0$$

$$= \frac{1}{2}\left[(1-E)^{-1} + (2+\Delta)^{-1}\right]u_0$$

$$= \frac{1}{2}\left[(1-E)^{-1} + (1+E)^{-1}\right]u_0$$

$$= \frac{1}{2}\cdot 2\left[1+E^2+E^4+E^6+\cdots\right]u_0$$

$$= u_0+u_2+u_4+u_6+\cdots$$

$$= \sum_{x=0}^{\infty} u_{2x} = \text{L.H.S.} \quad //$$

Ex) Prove that $\sum_{k=0}^{m-1} \Delta^2 f_k = \Delta f_m - \Delta f_0$

Solⁿ:- we have,

$$\Delta^2 f_k = (E-1)^2 f_k$$

$$= (E^2 - 2E + 1) f_k$$

$$= f_{k+2} - 2f_{k+1} + f_k$$

Now, $\sum_{k=0}^{m-1} \Delta^2 f_k = (f_2 - 2f_1 + f_0) + (f_3 - 2f_2 + f_1) + (f_4 - 2f_3 + f_2)$

$$+ \cdots + (f_{m+1} - 2f_m + f_{m-1})$$

$$= f_0 + (-2f_1 + f_1) + (f_2 - 2f_2 + f_2) + \cdots$$

$$+ (f_m - 2f_m) + f_{m+1}$$

$$= f_0 - f_1 - f_m + f_{m+1}$$

$$= (f_{m+1} - f_m) - (f_1 - f_0)$$

$$= \Delta f_m - \Delta f_0 \quad //$$