Theorem: The number $\sqrt{2}$ is irrational

Pf. If possible let $\sqrt{2} = \frac{a}{b}$ with $\gcd(a, b) = 1$, $b \neq 0$.

Since $\gcd(a, b) = 1$, so $\exists$ integers r and s such that $ar + bs = 1$.

Now, $\sqrt{2} = \sqrt{2} \cdot 1 = \sqrt{2}\,(ar + bs)$

$\Rightarrow \sqrt{2} = (\sqrt{2}\,a)\,r + (\sqrt{2}\,b)\,s$

$\Rightarrow \sqrt{2} = 2br + as$

$\sqrt{2} = \frac{a}{b}$

$\Rightarrow \sqrt{2}\,b = a$

also

$\sqrt{2} \cdot a = \frac{a^2}{b}$

$= \frac{2b^2}{b}$

$= 2b$

But as a, b, r and s are integers so $2br + as$ ~~are~~ is also an integer

$\Rightarrow \sqrt{2}$ is an integer, which is a contradiction.

Hence $\sqrt{2}$ cannot be expressed as $\frac{a}{b}$. Hence $\sqrt{2}$ is an irrational.

#

Ex. Prove that every integer $n \geq 2$ has a prime factor.

Soln. Pf. of fundamental Thm of arithmetic. No need to give the uniqueness part.

Scanned with CamScanner

Ex: For any integer $n \geq 2$ if p divides $a_1 \cdot a_2 \cdots a_n$ then prove that p divides one of the integers $a_1, a_2, \cdots, a_n$, where p is prime number. Applying this result, show that 12 is not a prime number.

Sol: First part already proved.

Second part: $12 \mid 3 \cdot 4$ but neither $12 \mid 3$ nor $12 \mid 4$. Hence 12 is not a prime number.

Ex. If n is a positive integer such that $n^3 + 1$ is a prime, then find the value of n.

Sol: Let $n^3 + 1 = p$, p is a prime

$$\Rightarrow (n+1)(n^2 - n + 1) = p.$$

Since p is a prime so p has only two divisors 1 and p.

$\therefore$ either $n+1 = 1$, $n^2 - n + 1 = p$ or $n+1 = p$, $n^2 - n + 1 = 1$

But $n+1 = 1$ is not possible as n is positive integer

$\therefore$ $n+1 = p$, $n^2 - n + 1 = 1 \Rightarrow n^2 - n = 0$

$\Rightarrow n(n-1) = 0 \Rightarrow n = 0$ or 1

But n cannot be 0. So n must be 1.

$\therefore$ $p = n+1 = 1+1 = 2$

#

Scanned with CamScanner

Ex: prove that the only prime of the form $n^3 - 1$ is 7.

Sol^n: Let $n^3 - 1 = p$, p is prime

$\Rightarrow (n-1)(n^2+n+1) = p.$

$\therefore$ either

$n-1 = p$, $n^2+n+1 = 1$ or $n-1 = 1$, $n^2+n+1 = p$.

But $n^2+n+1 = 1$ is not possible as n is positive integer.

$\therefore$ $n-1 = 1$, $n^2+n+1 = p$

$\Rightarrow n = 2$, $2^2+2+1 = p$

$\Rightarrow p = 7$

#

Ex

Scanned with CamScanner

Theorem: There is an infinite number of preimes. [Euclid].

Soln: If possible let there are only finite number of preimes say $p_1, p_2, \dots, p_k$.

Take $N = p_1 \cdot p_2 \cdot \dots p_k + 1$

Clearly $N > 1$, so by fundamental theorem of arithmetic N has a prime factor say p. i.e. $p \mid N = p_1 \cdot p_2 \cdots p_k + 1$ — (1)

Since there are only finite number of preimes so p is one of the preimes $p_1, p_2, \dots p_k$. Let $p = p_i$ for some i, $1 \le i \le k$.

Then $p \mid p_i \Rightarrow p \mid p_1 \cdot p_2 \dots p_k$.

~~Now~~, $\Rightarrow p_1 p_2 \dots p_k = px$, for some integer x.

$\therefore$ (1) $\Rightarrow p \mid p_1 p_2 \dots p_k + 1$

$\Rightarrow p_1 . p_2 \dots p_k + 1 = py$, for some integer y

$\Rightarrow px + 1 = py$

$\Rightarrow p(x - y) = 1 \Rightarrow p \mid 1$, which is not possible as p is prime. So there cannot be finite number of preimes.

Hence there is an infinite number of primes.