Permutation: Permutation of a set is a bijection from the set into itself. $f : [n] \longrightarrow [n]$ satisfying the condition $i \neq j$ implies $f(i) \neq f(j)$.

consider $f(i) = x_i$, then we can write

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$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{pmatrix}$$

eg. $n = 5$. $[n] = \{1, 2, 3, 4, 5\}$.

$f(1) = 2$, $f(2) = 4$, $f(3) = 3$, $f(4) = 5$, $f(5) = 1$.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

If n is known (which is generally the case), then the first line is redundant. It is therefore more common to use just the one line representation of a permutation:

$$x_1 x_2 \cdots x_{n-1} x_n.$$

$n = 5$ one line representation.
of

2 4 3 5 1.

it means: f is permutation with $f(5) = 1$, $f(1) = 2$
$f(2) = 4$, $f(4) = 3$, $f(3) = 5$

Definition: Let $\pi = x_1 x_2 \cdots x_{n-1} x_n$ be a permutation of the set $[n]$. A pair (k, m) with $k > m$ is called an inversion of π if for some i and some j with $i < j$, we have $x_i = k$, and $x_j = m$.

eg. $n = 5$, and $\pi = 24351$

pair of inversions are $(2,1)$, $(4,1)$, $(3,1)$, $(5,1)$ $(4,3)$

Observe: There is no inversions that ends in 5. none end in 4, one end in 3 | none end in 2, 4 end in 1.

$(0, 0, 1, 0, 4)$. $\longrightarrow$ it is called inversion sequence.

Inversion seq^n: Let $\pi = x_1, x_2, \ldots x_n$ be a permutation on $[n]$.
For $m = 1, 2, \ldots n$, let $b_m = b_m(\pi)$ denotes the number of inversions that ends in m, i.e. those inversions (k, m) when $k > m$.
The sequence $(b_1, b_2, \ldots b_{n-1}, b_n)$ is called inversion sequence of π, and is denoted by $Inv(\pi)$.

H.W

$n = 7$, and $\tau = 5123764$.

find inversions ?

and inversions sequence ?

Q Show that total no. of inversion sequence is atmost $n!$

$\Rightarrow$ Let π be ~~inversion~~ permutation on $[n]$.

We observe that.

no of inversion ends in n is 0, i.e $b_n = 0$

no of " " " $n-1$ is atmost 1 i.e $b_{n-1} \leq 1$.

In general,

no of inversion ends in k is atmost $n-k+1$

i.e $b_k \leq n-k+1$

Total no of inversion sequence is atmost.

$$\prod_{k=1}^{k=n} (n-k+1) = \prod_{r=0}^{r=n-1} (r+1) = n!$$

where $r = n-k$. $\quad n-k=r$

Thm: 2.15 Given a sequence $(b_1, b_2, \ldots, b_{n-1}, b_n)$ satisfying $0 \le b_k \le n-k$, for all k, there exists a unique permutation π such that

$$\mathrm{Inv}(\pi) = (b_1, b_2, \ldots, b_{n-1}, b_n).$$

Pf: The proof is constructive. ~~Suppose~~
Consider the given sequence $(b_1, b_2, \ldots b_{n-1}, b_n)$ satisfying $0 \le b_k \le n-k$.

The ~~proof~~ ~~proceeds in steps, partially completing the~~ ~~Required permutation~~

Fix a k between 1 and n. We follow an insertion procedure, the integer k being inserted at step $(n+1-k)$.

Step (1) write $\bar{\pi}_1 = n$.

Step (2) Write $\bar{\pi}_2 = n-1, n$ if $b_{n-1} = 0$

and $\bar{\pi}_2 = n, n-1$ if $b_{n-1} = 1$

At step $n-k$, we would have obtained $\bar{\pi}_{n-k}$ which is a partial permutation consisting of the integers from $k+1$ to n.

Step $n+1-k$; We have to insert k in the partial permutation $\bar{\pi}_{n-k}$. We insert k after the first b_k integers in the partial permutation $\bar{\pi}_{n-k}$.

Since at any further steps, the relative order of the integers $k, k+1, \ldots n$ is not be disturbed, it follows that the final permutation $\pi_n = \bar{\pi}$, the value $b_k(\bar{\pi})$ equals b_k.

Thus step n completes the construction of $\bar{\pi}$.