※ The solution of exact differential equation:

If the equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact in a rectangular domain D, then there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y) \text{ for all } (x,y) \in D.$$

Then the equation may be written as

$$\frac{\partial F(x,y)}{\partial x}\,dx + \frac{\partial F(x,y)}{\partial y}\,dy = 0 \quad \text{or simply} \quad dF(x,y) = 0$$

The relation $F(x,y) = c$ is obviously a solution of this, where c is an arbitrary constant. We summarize this observation in the following theorem.

* Suppose the differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ satisfies the differentiability requirements of the Previous theorem and is exact in a rectangular domain D. Then a one-parameter family of solutions of this differential equation is given by $F(x,y) = c$, where F is a function such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y) \text{ for all } (x,y) \in D$$

and c is an arbitrary constant.

Ex) Solve the equation:

$$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$$

Soln: Here

$$M(x,y) = 3x^2 + 4xy$$

$$N(x,y) = 2x^2 + 2y$$

Now $\frac{\partial M(x,y)}{\partial y} = 4x$ and $\frac{\partial N(x,y)}{\partial x} = 4x$ for all real (x,y), and so the equation is exact in every rectangular domain D. Thus we must find F such that $\frac{\partial F(x,y)}{\partial x} = M(x,y) = 3x^2 + 4xy$ and $\frac{\partial F(x,y)}{\partial y} = N(x,y) = 2x^2 + 2y$.

From the first of these,

$$F(x,y) = \int M(x,y)\,\partial x + \phi(y)$$

$$= \int (3x^2 + 4xy)\,\partial x + \phi(y) = x^3 + 2x^2 y + \phi(y)$$

$$\therefore \quad \frac{\partial F(x,y)}{\partial y} = 2x^2 + \frac{d\phi(y)}{dy}$$

But we must have,

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = 2x^2 + 2y$$

$$\Rightarrow \quad 2x^2 + \frac{d\phi(y)}{dy} = 2x^2 + 2y$$

$$\Rightarrow \quad \frac{d\phi(y)}{dy} = 2y$$

Thus $\phi(y) = y^2 + c_0$, where c_0 is an arbitrary constant and so $F(x,y) = x^3 + 2x^2 y + y^2 + c_0$

Hence a one-parameter family of solutions is

$$F(x,y) = c_1$$

$$\Rightarrow \quad x^3 + 2x^2 y + y^2 + c_0 = c_1$$

Combining the constants c_0 and c_1, we may write the solution as

$$x^3 + 2x^2 y + y^2 = c$$

where $c = c_1 - c_0$ is an arbitrary constant.