Example: The differential eqⁿ

$$y^2\,dx + 2xy\,dy = 0 \quad \text{-----}(1)$$

is an exact differential equation, since the expression $y^2 dx + 2xy\,dy$ is an exact differential. Indeed, it is the total differential of the function F defined for all (x,y) by $F(x,y) = xy^2$, since the coefficient of dx is $\frac{\partial F(x,y)}{\partial x} = y^2$ and that of dy is $\frac{\partial F(x,y)}{\partial y} = 2xy$. On the other hand, the more simple appearing equation

$$y\,dx + 2x\,dy = 0 \quad \text{-----}(2)$$

obtained from (1) by dividing through by y, is not exact.

Theorem:

Consider the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{-----}(3)$$

where M and N have continuous first partial derivatives at all points (x,y) in a rectangular domain D.

(1) If the differential eqⁿ (3) is exact in D, then

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \quad \text{-----}(4)$$

for all $(x,y) \in D$

(2) Conversely, if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

for all $(x,y) \in D$, then the differential equation (3) is exact in D.

Example:

For the equation

$$y^2 dx + 2xy\, dy = 0$$

we have,

$$M(x,y) = y^2, \quad N(x,y) = 2xy$$

$$\therefore \frac{\partial M(x,y)}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N(x,y)}{\partial x} = 2y$$

$$\therefore \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \quad \text{for all } (x,y).$$

Thus equation $y^2 dx + 2xy\, dy = 0$ is exact in every rectangular domain D. On the other hand, for the equation,

$$y\, dx + 2x\, dy = 0$$

we have,

$$M(x,y) = y, \quad N(x,y) = 2x$$

$$\therefore \frac{\partial M(x,y)}{\partial y} = 1 \quad \text{and} \quad \frac{\partial N(x,y)}{\partial x} = 2$$

$$\therefore \frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$$ for all (x,y). Thus equation $y\, dx + 2x\, dy = 0$ is not exact in any rectangular domain D.

Consider the differential equation

$$(2x \sin y + y^3 e^x)\, dx + (x^2 \cos y + 3y^2 e^x)\, dy = 0$$

Here,

$$M(x,y) = 2x \sin y + y^3 e^x$$

$$N(x,y) = x^2 \cos y + 3y^2 e^x$$

Now,

$$\frac{\partial M(x,y)}{\partial y} = 2x \cos y + 3y^2 e^x$$

$$\frac{\partial N(x,y)}{\partial x} = 2x \cos y + 3y^2 e^x$$

$$\therefore \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$ in every rectangular domain D. Thus this differential equation is exact in every such domain.