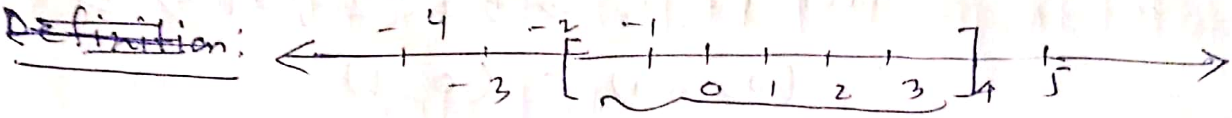
Meaning of $|a-b|$

* $|a-b|$ denotes the distance between the real numbers a and b.

$|a| = |a-0|$ denotes the distance between the real numbers a and 0.

* If a is a given real number, then saying that a real number x is "close to" a should mean that the distance $|x-a|$ between them is small.

~~Definition~~:

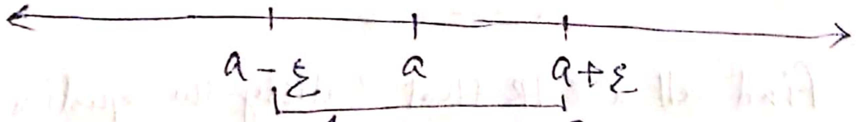


$$|(-2) - 4| = |-2-4| = |-6| = 6$$

The distance between $a = 2$ and $b = 4$.

Definition: Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the ε-neighbourhood (nbhd) of a is the set $V_\varepsilon(a) = \{x \in \mathbb{R} : |x-a| < \varepsilon\}$

$$\begin{aligned} V_\varepsilon(a) &= \{x \in \mathbb{R} ; |x-a| < \varepsilon\} \\ &= \{x \in \mathbb{R} : -\varepsilon < x-a < \varepsilon\} \\ &= \{x \in \mathbb{R} ; a-\varepsilon < x < a+\varepsilon\} \\ &= (a-\varepsilon, a+\varepsilon). \end{aligned}$$



An ε-nbhd of a.

Theorem: Let $a \in \mathbb{R}$. If x belongs to the neighbourhood $V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.

Pf:- Since $x \in V_\varepsilon(a)$ for every $\varepsilon > 0$

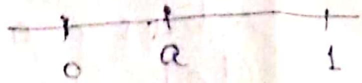
$\therefore$ $|x-a| < \varepsilon$, for every $\varepsilon > 0$.

But $|x-a| \geq 0$. So we have seen that a non negative number is less than every positive number.

Hence $|x-a| = 0 \Rightarrow x - a = 0 \Rightarrow x = a$.

Ex. Let $U=\{x: 0<x<1\}$. If $a\in U$, let $\varepsilon=\min(a,1-a)$
Show that $V_\varepsilon(a)\subseteq U$.

Soln. $U=\{x: 0<x<1\}=(0,1)$



$V_\varepsilon(a)=\{x\in\mathbb{R}: |x-a|<\varepsilon\}$

Case 1. Let $\min(a,1-a)=a$. Then $\varepsilon=a$ and $a<\frac{1}{2}$

Then $V_\varepsilon(a)=\{x\in\mathbb{R}: |x-a|<\varepsilon\}=(a-\varepsilon, a+\varepsilon)$

$=(a-a, a+a)=(0,2a)$

Since $a<\frac{1}{2} \Rightarrow 2a<2\cdot\frac{1}{2}=1 \Rightarrow 2a<1$.

$\therefore (0,2a)\subseteq(0,1)$

$\Rightarrow V_\varepsilon(a)\subseteq U$.

Case 2. Let $\min(a,1-a)=1-a$. Then $a>\frac{1}{2}$
and $\varepsilon=1-a$.

$V_\varepsilon(a)=(a-\varepsilon, a+\varepsilon)=(a-(1-a), a+1-a)$

$=(2a-1, 1)$

Since $a>\frac{1}{2} \Rightarrow 2a>2\cdot\frac{1}{2}=1 \Rightarrow 2a>1 \Rightarrow 2a-1>0$.

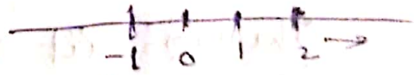
Also $a<1 \Rightarrow 2a<2 \Rightarrow 2a-1<2-1 \Rightarrow 2a-1<1$.

$\therefore (2a-1, 1)\subseteq(0,1)$

$\Rightarrow V_\varepsilon(a)\subseteq U$.

Ex. Find all $x\in\mathbb{R}$ that satisfy the equation $|x+1|+|x-2|=7$.

Soln. Case 1. For $x\geq 2$. $|x-2|=x-2$
and $|x+1|=x+1$



$\therefore |x+1|+|x-2|=7 \Rightarrow x+1+x-2=7 \Rightarrow 2x=8 \Rightarrow x=4$.

Case 2. For $-1\leq x<2$. Then $|x-2|=-(x-2)$ and $|x+1|=x+1$

$\therefore |x+1|+|x-2|=7 \Rightarrow x+1-x+2=7 \Rightarrow 3=4$, which is an absurd result.

Case 3. For $x<-1$. Then $|x+1|=-(x+1)$, $|x-2|=-(x-2)$
$=-x+2$

$\therefore |x+1|+|x-2|=7 \Rightarrow -(x+1)-x+2=7$
$\Rightarrow -2x=6 \Rightarrow x=-3$

$\therefore x=4$ or $x=-3$ will satisfy the equation $|x+1|+|x-2|=7$

(HW) ① If $a, b \in \mathbb{R}$ and $b \neq 0$, show that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

② Find all $x \in \mathbb{R}$ that satisfy the following inequalities

ⓐ $|4x-5| \leq 13$ ⓑ $|x| + |x+1| < 2$.

ⓒ $4 < |x+2| + |x-1| < 5$.

12. Determine and sketch the set of pairs (x, y) in $\mathbb{R} \times \mathbb{R}$ that satisfy

ⓐ $|x| = |y|$, ⓑ $|x| + |y| = 1$, ⓒ $|x| - |y| \geq 2$.

Soln.

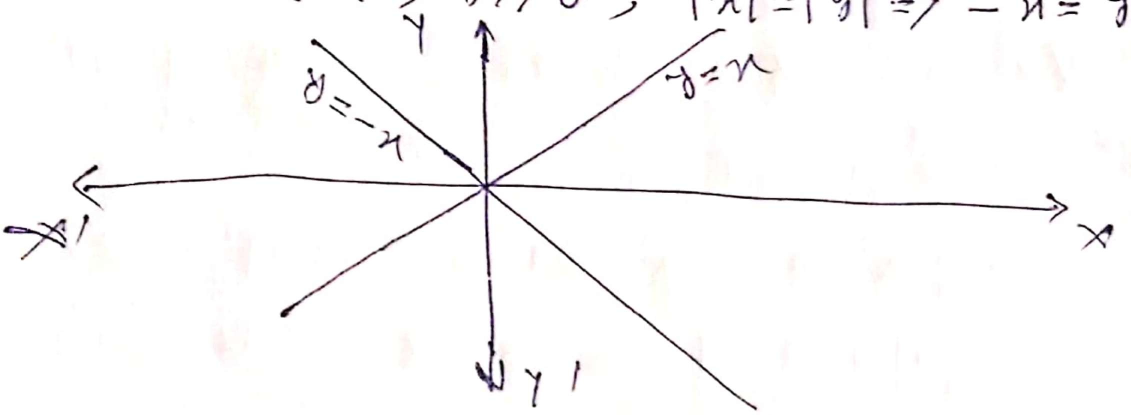
ⓐ

$$|x| = |y|$$

Case 1. for $x \geq 0$, $y \geq 0$, $|x| = |y| \Rightarrow x = y \Rightarrow y = x$

Case 2. for $x \geq 0$, $y < 0$, $|x| = |y| \Rightarrow x = -y \Rightarrow y = -x$

Case 3. for $x < 0$, $y \geq 0$, $|x| = |y| \Rightarrow -x = y \Rightarrow y = -x$



ⓑ and (c) are homeworks.