49. If $f(x, y)$ is continuous for all $x \neq 0$ and $y \neq 0$, and $f(0, 0) = 0$, then $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

50. If $f(x)$ and $g(y)$ are continuous functions of x and y, respectively, then

$$h(x, y) = f(x) + g(y)$$

is a continuous function of x and y.

Use the ϵ-δ definition of limit to verify the limit statements given in Problems 51–54.

51. $\lim_{(x,y)\to(0,0)} (2x^2 + 3y^2) = 0$

52. $\lim_{(x,y)\to(0,0)} (x + y^2) = 0$

53. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x + y} = 0$

54. $\lim_{(x,y)\to(1,-1)} \dfrac{x^2 - y^2}{x + y} = 2$

55. Prove that if f is continuous and $f(a, b) > 0$, then there exists a δ-neighborhood about (a, b) such that $f(x, y) > 0$ for every point (x, y) in the neighborhood.

56. Prove the scalar multiple rule:

$$\lim_{(x,y)\to(x_0,y_0)} [af](x, y) = a \lim_{(x,y)\to(x_0,y_0)} f(x, y)$$

57. Prove the sum rule:

$$\lim_{(x,y)\to(x_0,y_0)} [f + g](x, y) = L + M$$

where $L = \lim_{(x,y)\to(x_0,y_0)} f(x, y)$ and $M = \lim_{(x,y)\to(x_0,y_0)} g(x, y)$.

58. A function of two variables $f(x, y)$ may be continuous in each separate variable at $x = x_0$ and $y = y_0$ without being itself continuous at (x_0, y_0). Let $f(x, y)$ be defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$

Let $g(x) = f(x, 0)$ and $h(y) = f(0, y)$. Show that both $g(x)$ and $h(y)$ are continuous at 0 but that $f(x, y)$ is not continuous at $(0, 0)$.

11.3 Partial Derivatives

IN THIS SECTION partial differentiation, partial derivative as a slope, partial derivative as a rate, higher-order partial derivatives

PARTIAL DIFFERENTIATION

It is often important to know how a function of two variables changes with respect to one of the variables. For example, according to the ideal gas law, the pressure of a gas is related to its temperature and volume by the formula $P = \dfrac{kT}{V}$, where k is a constant. If the temperature is kept constant while the volume is allowed to vary, we might want to know the effect on the rate of change of pressure. Similarly, if the volume is kept constant while the temperature is allowed to vary, we might want to know the effect on the rate of change of pressure.

The process of differentiating a function of several variables with respect to one of its variables while keeping the other variable(s) fixed is called **partial differentiation**, and the resulting derivative is a **partial derivative** of the function.

Recall that the derivative of a function of a single variable f is defined to be the limit of a difference quotient, namely,

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Partial derivatives with respect to x or y are defined similarly.

Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$, then the partial derivatives of f with respect to x and y are the functions f_x and f_y, respectively, defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limits exist.

➜ What This Says For the partial differentiation of a function of two variables, $z = f(x, y)$, we find the partial derivative with respect to x by regarding y as constant while differentiating the function with respect to x. Similarly, the partial derivative with respect to y is found by regarding x as constant while differentiating with respect to y.

EXAMPLE 1 Partial derivatives

If $f(x, y) = x^3y + x^2y^2$, find: **a.** f_x **b.** f_y

Solution

a. For f_x, hold y constant and find the derivative with respect to x:

$$f_x(x, y) = 3x^2y + 2xy^2$$

b. For f_y, hold x constant and find the derivative with respect to y:

$$f_y(x, y) = x^3 + 2x^2y$$ ■

Several different symbols are used to denote partial derivatives, as indicated in the following box.

Alternative Notation for Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = z_x = D_x(f)$$

and

$$f_y(x, y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = z_y = D_y(f)$$

The values of the partial derivatives of $f(x, y)$ at the point (a, b) are denoted by

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a, b)$$

EXAMPLE 2 Finding and evaluating a partial derivative

Let $z = x^2 \sin(3x + y^3)$.

a. Evaluate $\left.\dfrac{\partial z}{\partial x}\right|_{(\pi/3,0)}$. **b.** Evaluate z_y at $(1, 1)$.

Solution

a. $$\frac{\partial z}{\partial x} = 2x\sin(3x + y^3) + x^2\cos(3x + y^3)(3)$$
$$= 2x\sin(3x + y^3) + 3x^2\cos(3x + y^3)$$

Thus,

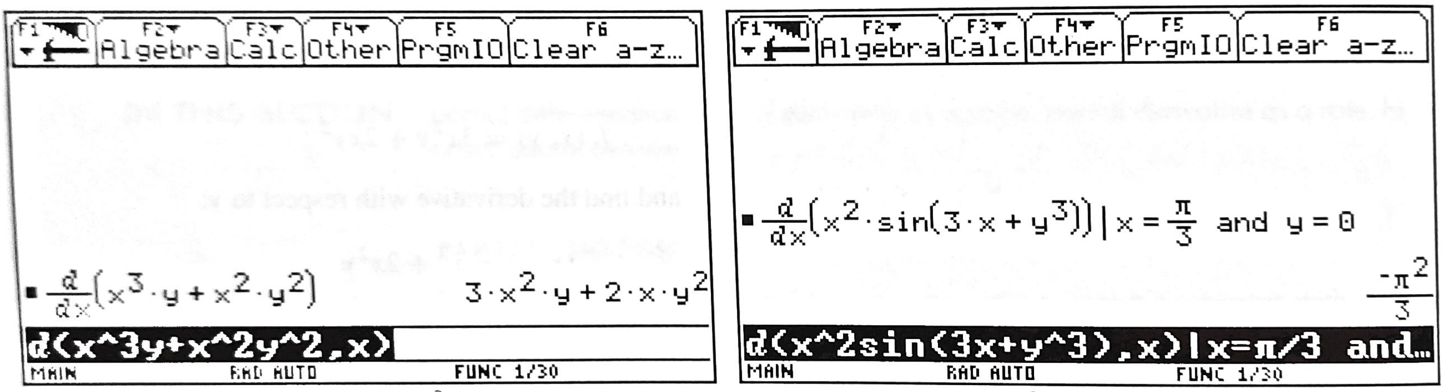
$$\left.\frac{\partial z}{\partial x}\right|_{(\pi/3,0)} = 2\left(\frac{\pi}{3}\right)\sin\pi + 3\left(\frac{\pi}{3}\right)^2\cos\pi = \frac{2\pi}{3}(0) + \frac{\pi^2}{3}(-1) = -\frac{\pi^2}{3}$$

b. $z_y = x^2\cos(3x + y^3)(3y^2) = 3x^2y^2\cos(3x + y^3)$ so that

$$z_y(1, 1) = 3(1)^2(1)^2\cos(3 + 1) = 3\cos 4$$

TECHNOLOGY NOTE

Finding partial derivatives using technology is a natural extension of the way you have been finding other derivatives. The general format for most calculators and computer programs is the same: *derivative operator, function, variable of differentiation.* Evaluating the partial derivative is then accomplished by using the evaluate feature. For example, Figure 11.17a displays the computation of the partial derivative of $f(x, y) = x^3y + x^2y^2$ with respect to x from Example 1, while Figure 11.17b displays the evaluation of the partial derivative of $z = x^2\sin(3x + y^3)$ with respect to x at the point $(\frac{\pi}{3}, 0)$ found in Example 2.



a. The partial derivative $\frac{\partial}{\partial x}(x^3y + x^2y^2)$ **b.** The partial derivative $\frac{\partial}{\partial x}(x^2\sin(3x + y^3))$ evaluated at $(\frac{\pi}{3}, 0)$

Figure 11.17 Computing partial derivatives with technology

EXAMPLE 3 Partial derivative of a function of three variables

Let $f(x, y, z) = x^2 + 2xy^2 + yz^3$; determine: **a.** f_x **b.** f_y **c.** f_z

Solution

a. For f_x, think of f as a function of x alone with y and z treated as constants:

$$f_x(x, y, z) = 2x + 2y^2$$

b. $f_y(x, y, z) = 4xy + z^3$

c. $f_z(x, y, z) = 3yz^2$

WARNING In $f(x, y, z)$, z is an independent variable.

EXAMPLE 4 Partial derivative of an implicitly defined function

Let z be defined implicitly as a function of x and y by the equation

$$x^2z + yz^3 = x$$

Determine $\partial z/\partial x$ and $\partial z/\partial y$.

Solution

Differentiate implicitly with respect to x, treating y as a constant:

$$2xz + x^2\frac{\partial z}{\partial x} + 3yz^2\frac{\partial z}{\partial x} = 1$$

Then solve this equation for $\frac{\partial z}{\partial x}$:

$$\frac{\partial z}{\partial x} = \frac{1 - 2xz}{x^2 + 3yz^2}$$

Similarly, holding x constant and differentiating implicitly with respect to y, we find

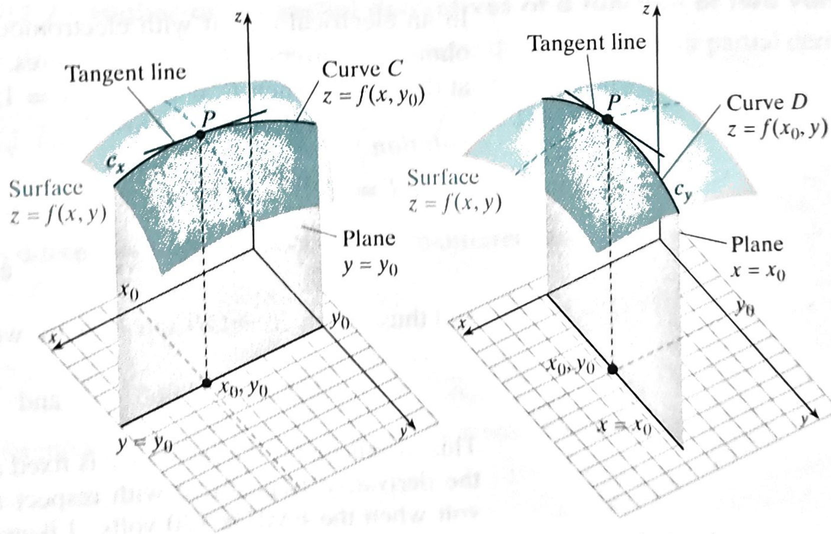
$$x^2\frac{\partial z}{\partial y} + z^3 + 3yz^2\frac{\partial z}{\partial y} = 0$$

so that

$$\frac{\partial z}{\partial y} = \frac{-z^3}{x^2 + 3yz^2}$$ ■

PARTIAL DERIVATIVE AS A SLOPE

A useful geometric interpretation of partial derivatives is indicated in Figure 11.18. In Figure 11.18a, the plane $y = y_0$ intersects the surface $z = f(x, y)$ in a curve C parallel to the xz-plane. That is, C is the trace of the surface in the plane $y = y_0$. An equation for this curve is $z = f(x, y_0)$, and because y_0 is fixed, the function depends only on x. Thus, we can compute the slope of the tangent line to C at the point $P(x_0, y_0, z_0)$ in the plane $y = y_0$ by differentiating $f(x, y_0)$ with respect to x and evaluating the derivative at $x = x_0$. That is, the slope is $f_x(x_0, y_0)$, the value of the partial derivative f_x at (x_0, y_0). The analogous interpretation for $f_y(x_0, y_0)$ is shown in Figure 11.18b.



a. The tangent line in the plane $y = y_0$ to the curve C at the point P has slope $f_x(x_0, y_0)$.

b. The tangent line in the plane $x = x_0$ to the curve D at the point P has slope $f_y(x_0, y_0)$.

Figure 11.18 Slope interpretation of the partial derivative

Partial Derivative as the Slope of a Tangent Line

The line parallel to the xz-plane and tangent to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0)$ has slope $f_x(x_0, y_0)$. Likewise, the tangent line to the surface at P_0 that is parallel to the yz-plane has slope $f_y(x_0, y_0)$.

EXAMPLE 5 Slope of a line parallel to the xz-plane

Find the slope of the line that is parallel to the xz-plane and tangent to the surface $z = x\sqrt{x+y}$ at the point $P(1, 3, 2)$.

Solution

If $f(x, y) = x\sqrt{x+y} = x(x+y)^{1/2}$, then the required slope is $f_x(1, 3)$.

$$f_x(x, y) = x\left(\frac{1}{2}\right)(x+y)^{-1/2}(1+0) + (1)(x+y)^{1/2} = \frac{x}{2\sqrt{x+y}} + \sqrt{x+y}$$

Thus, $f_x(1, 3) = \dfrac{1}{2\sqrt{1+3}} + \sqrt{1+3} = \dfrac{9}{4}$. ■

PARTIAL DERIVATIVE AS A RATE

The derivative of a function of one variable can be interpreted as a rate of change, and the analogous interpretation of partial derivative may be described as follows.

> **Partial Derivatives as Rates of Change**
>
> As the point (x, y) moves from the fixed point $P_0(x_0, y_0)$, the function $f(x, y)$ changes at a rate given by $f_x(x_0, y_0)$ in the direction of the positive x-axis and by $f_y(x_0, y_0)$ in the direction of the positive y-axis.

EXAMPLE 6 Partial derivatives as rates of change

In an electrical circuit with electromotive force (EMF) of E volts and resistance R ohms, the current is $I = E/R$ amperes. Find the partial derivatives $\partial I/\partial E$ and $\partial I/\partial R$ at the instant when $E = 120$ and $R = 15$ and interpret these derivatives as rates.

Solution

Since $I = ER^{-1}$, we have

$$\frac{\partial I}{\partial E} = R^{-1} \quad \text{and} \quad \frac{\partial I}{\partial R} = -ER^{-2}$$

and thus, when $E = 120$ and $R = 15$, we find that

$$\frac{\partial I}{\partial E} = 15^{-1} \approx 0.0667 \quad \text{and} \quad \frac{\partial I}{\partial R} = -(120)(15)^{-2} \approx -0.5333$$

This means that if the resistance is fixed at 15 ohms, the current is increasing (because the derivative is positive) with respect to voltage at the rate of 0.0667 ampere per volt when the EMF is 120 volts. Likewise, with the same fixed EMF, the current is decreasing (because the derivative is negative) with respect to resistance at the rate of 0.5333 ampere per ohm when the resistance is 15 ohms. ■

HIGHER-ORDER PARTIAL DERIVATIVES

The partial derivative of a function is a function, so it is possible to take the partial derivative of a partial derivative. This is very much like taking the second derivative of a function of one variable if we take two consecutive partial derivatives with respect to the same variable, and the resulting derivative is called the **second-order partial derivative** with respect to that variable. However, we can also take the partial derivative with respect to one variable and then take a second partial derivative with respect to a different variable, producing what is called a **mixed second-order partial derivative**. The higher-order partial derivatives for a function of two variables $f(x, y)$ are denoted as indicated in the following box:

Higher-Order Partial Derivatives

Given $z = f(x, y)$.

Second-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy}$$

Mixed second-order partial derivatives

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$$

WARNING The notation f_{xy} means that we differentiate first with respect to x and then with respect to y, while $\frac{\partial^2 f}{\partial x \partial y}$ means just the opposite (differentiate with respect to y first and then with respect to x).

EXAMPLE 7 Higher-order partial derivatives of a function of two variables

For $z = f(x, y) = 5x^2 - 2xy + 3y^3$, determine these higher-order partial derivatives.

a. $\frac{\partial^2 z}{\partial x \partial y}$ **b.** $\frac{\partial^2 f}{\partial y \partial x}$ **c.** $\frac{\partial^2 z}{\partial x^2}$ **d.** $f_{xy}(3, 2)$

Solution

a. First differentiate with respect to y; then differentiate with respect to x.

$$\frac{\partial z}{\partial y} = -2x + 9y^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-2x + 9y^2) = -2$$

b. Differentiate first with respect to x and then with respect to y:

$$\frac{\partial f}{\partial x} = 10x - 2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(10x - 2y) = -2$$

c. Differentiate with respect to x twice:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(10x - 2y) = 10$$

d. Evaluate the mixed partial found in part **b** at the point $(3, 2)$:

$$f_{xy}(3, 2) = -2$$

Notice from parts **a** and **b** of Example 7 that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$. This equality of mixed partials does not hold for all functions, but for most functions we will encounter, it will be true. The following theorem provides sufficient conditions for this equality to occur.

THEOREM 11.1 Equality of mixed partials

If the function $f(x, y)$ has mixed second-order partial derivatives f_{xy} and f_{yx} that are continuous in an open disk containing (x_0, y_0), then

$$f_{yx}(x_0, y_0) = f_{xy}(x_0, y_0)$$

Proof This proof is omitted. ❑

EXAMPLE 8 Partial derivatives of functions of two variables

Determine f_{xy}, f_{yx}, f_{xx}, and f_{xxy}, where $f(x, y) = x^2ye^y$.

Solution

We have the partial derivatives

$$f_x = 2xye^y \qquad f_y = x^2e^y + x^2ye^y$$

The mixed partial derivatives (which must be the same by the previous theorem) are

$$f_{xy} = (f_x)_y = 2xe^y + 2xye^y \qquad f_{yx} = (f_y)_x = 2xe^y + 2xye^y$$

Finally, we compute the second- and higher-order partial derivatives:

$$f_{xx} = (f_x)_x = 2ye^y \quad \text{and} \quad f_{xxy} = (f_{xx})_y = 2e^y + 2ye^y$$ ■

An equation involving partial derivatives is called a **partial differential equation.** An important partial differential equation is the *diffusion* or *heat equation*

$$\frac{\partial T}{\partial t} = c^2\frac{\partial^2 T}{\partial x^2}$$

where $T(x, t)$ is the temperature in a thin rod at position x and time t. The constant c is called the *diffusivity* of the material in the rod. In the following example, we verify that a certain function satisfies this heat equation.

EXAMPLE 9 Verifying that a function satisfies the heat equation

Verify that $T(x, t) = e^{-t}\cos\frac{x}{c}$ satisfies the heat equation, $\frac{\partial T}{\partial t} = c^2\frac{\partial^2 T}{\partial x^2}$.

Solution

$$\frac{\partial T}{\partial t} = -e^{-t}\cos\frac{x}{c}$$

and

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{1}{c}e^{-t}\sin\frac{x}{c}\right) = -\frac{1}{c^2}e^{-t}\cos\frac{x}{c}$$

Thus, T satisfies the heat equation $\frac{\partial T}{\partial t} = c^2\frac{\partial^2 T}{\partial x^2}$. ■

Analogous definitions can be made for functions of more than two variables. For example,

$$f_{zzz} = \frac{\partial^3 f}{\partial z^3} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)\right] \quad \text{or} \quad f_{xyz} = \frac{\partial^3 f}{\partial z\partial y\partial x} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right]$$

EXAMPLE 10 **Higher-order partial derivatives of a function of several variables**

By direct calculation, show that $f_{xyz} = f_{yzx} = f_{zyx}$ for the function $f(x, y, z) = xyz + x^2y^3z^4$.

Solution

First, compute the partials:

$$f_x(x, y, z) = yz + 2xy^3z^4$$
$$f_y(x, y, z) = xz + 3x^2y^2z^4$$
$$f_z(x, y, z) = xy + 4x^2y^3z^3$$

Next, determine the mixed partials:

$$f_{xy}(x, y, z) = (yz + 2xy^3z^4)_y = z + 6xy^2z^4$$
$$f_{yz}(x, y, z) = (xz + 3x^2y^2z^4)_z = x + 12x^2y^2z^3$$
$$f_{zy}(x, y, z) = (xy + 4x^2y^3z^3)_y = x + 12x^2y^2z^3$$

Finally, obtain the required higher mixed partials:

$$f_{xyz}(x, y, z) = (z + 6xy^2z^4)_z = 1 + 24xy^2z^3$$
$$f_{yzx}(x, y, z) = (x + 12x^2y^2z^3)_x = 1 + 24xy^2z^3$$
$$f_{zyx}(x, y, z) = (x + 12x^2y^2z^3)_x = 1 + 24xy^2z^3$$

11.3 PROBLEM SET

A 1. **WHAT DOES THIS SAY?** What is a partial derivative?

2. **Exploration Problem** Describe two fundamental interpretations of the partial derivatives $f_x(x, y)$ and $f_y(x, y)$.

Determine f_x, f_y, f_{xx}, and f_{yx} in Problems 3–8.

3. $f(x, y) = x^3 + x^2y + xy^2 + y^3$
4. $f(x, y) = (x + xy + y)^3$
5. $f(x, y) = \dfrac{x}{y}$
6. $f(x, y) = xe^{xy}$
7. $f(x, y) = \ln(2x + 3y)$
8. $f(x, y) = \sin x^2 y$

Determine f_x and f_y in Problems 9–16.

9. a. $f(x, y) = (\sin x^2)\cos y$ b. $f(x, y) = \sin(x^2 \cos y)$
10. a. $f(x, y) = (\sin\sqrt{x})\ln y^2$ b. $f(x, y) = \sin(\sqrt{x}\ln y^2)$
11. $f(x, y) = \sqrt{3x^2 + y^4}$
12. $f(x, y) = xy^2\ln(x + y)$
13. $f(x, y) = x^2e^{x+y}\cos y$
14. $f(x, y) = xy^3\tan^{-1} y$
15. $f(x, y) = \sin^{-1}(xy)$
16. $f(x, y) = \cos^{-1}(xy)$

Determine f_x, f_y, and f_z in Problems 17–22.

17. $f(x, y, z) = xy^2 + yz^3 + xyz$
18. $f(x, y, z) = xye^z$
19. $f(x, y, z) = \dfrac{x + y^2}{z}$
20. $f(x, y, z) = \dfrac{xy + yz}{xz}$
21. $f(x, y, z) = \ln(x + y^2 + z^3)$
22. $f(x, y, z) = \sin(xy + z)$

In Problems 23–28, determine $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ by differentiating implicitly.

23. $\dfrac{x^2}{9} - \dfrac{y^2}{4} + \dfrac{z^2}{2} = 1$
24. $3x^2 + 4y^2 + 2z^2 = 5$
25. $3x^2y + y^3z - z^2 = 1$
26. $x^3 - xy^2 + yz^2 - z^3 = 4$
27. $\sqrt{x} + y^2 + \sin xz = 2$
28. $\ln(xy + yz + xz) = 5$ $(x > 0, \quad y > 0, \quad z > 0)$

In Problems 29–32, compute the slope of the tangent line to the graph of f at the given point P_0 in the direction parallel to

a. the xz-plane *b. the yz-plane*

29. $f(x, y) = xy^3 + x^3y$; $P_0(1, -1, -2)$
30. $f(x, y) = \dfrac{x^2 + y^2}{xy}$; $P_0(1, -1, -2)$
31. $f(x, y) = x^2\sin(x + y)$; $P_0\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$
32. $f(x, y) = x\ln(x + y^2)$; $P_0(e, 0, e)$

B 33. Determine f_x and f_y for

$$f(x, y) = \int_x^y (t^2 + 2t + 1)\,dt$$

Hint: Review the second fundamental theorem of calculus.

34. Determine f_x and f_y for

$$f(x, y) = \int_{x^2}^{2y} (e^t + 3t)\, dt$$

Hint: Review the second fundamental theorem of calculus.

A function $f(x, y)$ is said to be **harmonic** *on the open set S if f_{xx} and f_{yy} are continuous and*

$$f_{xx} + f_{yy} = 0$$

throughout S. Show that each function in Problems 35–38 is harmonic on the given set.

35. $f(x, y) = 3x^2y - y^3$; S is the entire plane.
36. $f(x, y) = \ln(x^2 + y^2)$; S is the plane with the point $(0, 0)$ removed.
37. $f(x, y) = e^x \sin y$; S is the entire plane.
38. $f(x, y) = \sin x \cosh y$; S is the entire plane.
39. For $f(x, y) = \cos xy^2$, show $f_{xy} = f_{yx}$.
40. For $f(x, y) = (\sin^2 x)(\sin y)$, show $f_{xy} = f_{yx}$.
41. Find $f_{xzy} - f_{yzz}$, where $f(x, y, z) = x^2 + y^2 - 2xy \cos z$.
42. Two commodities Q_1 and Q_2 are said to be *substitute commodities* if an increase in the demand for either results in a decrease in the demand of the other. Let $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$ be the demand functions for Q_1 and Q_2, respectively, where p_1 and p_2 are the respective unit prices for the commodities.
    a. Explain why $\dfrac{\partial D_1}{\partial p_1} < 0$ and $\dfrac{\partial D_2}{\partial p_2} < 0$.
    b. Are $\dfrac{\partial D_1}{\partial p_2}$ and $\dfrac{\partial D_2}{\partial p_1}$ positive or negative? Explain.
    c. Give examples of substitute commodities.
43. Two commodities Q_1 and Q_2 are said to be *complementary commodities* if a decrease in the demand for either results in a decrease in the demand of the other. Let $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$ be the demand functions for Q_1 and Q_2, respectively, where p_1 and p_2 are the respective unit prices for the commodities.
    a. Is it true that $\dfrac{\partial D_1}{\partial p_1} < 0$ and $\dfrac{\partial D_2}{\partial p_2} < 0$? Explain.
    b. Determine whether $\dfrac{\partial D_1}{\partial p_2}$ and $\dfrac{\partial D_2}{\partial p_1}$ are positive or negative? Explain.
    c. Give examples of complementary commodities.
44. **Modeling Problem** The flow (in cm^3/s) of blood from an artery into a small capillary can be modeled by

$$F(x, y, z) = \frac{c\pi x^2}{4}\sqrt{y - z}$$

for constant $c > 0$, where x is the diameter of the capillary, y is the pressure in the artery, and z is the pressure in the capillary. Compute the rate of change of the flow of blood with respect to

a. the diameter of the capillary
b. the arterial pressure
c. the capillary pressure

45. **Modeling Problem** Biologists have studied the oxygen consumption of certain furry mammals. They have found that if the mammal's body temperature is T degrees Celsius, fur temperature is t degrees Celsius, and the mammal does not sweat, then its relative oxygen consumption can be modeled by

$$C(m, t, T) = \sigma(T - t)m^{-0.67}$$

(kg/h), where m is the mammal's mass (in kg) and σ is a physical constant. Compute the rate (rounded to two decimal places) at which the oxygen consumption changes with respect to

a. the mass m
b. the body temperature T
c. the fur temperature t

46. **Modeling Problem** A gas that gathers on a surface in a condensed layer is said to be *adsorbed* on the surface, and the surface is called an *adsorbing surface*. The amount of gas adsorbed per unit area on an adsorbing surface can be modeled by

$$S(p, T, h) = ape^{h/(bT)}$$

where p is the gas pressure, T is the temperature of the gas, h is the heat of the adsorbed layer of gas, and a and b are physical constants. Compute the rate of change of S with respect to

a. p b. h c. T

47. The *ideal gas law* says that $PV = kT$, where P is the pressure of a confined gas, V is the volume, T is the temperature, and k is a physical constant.
    a. Calculate $\dfrac{\partial V}{\partial T}$. b. Calculate $\dfrac{\partial P}{\partial V}$.
    c. Show that $\dfrac{\partial P}{\partial V} \cdot \dfrac{\partial V}{\partial T} \cdot \dfrac{\partial T}{\partial P} = -1$.
48. At a certain factory, the output is given by the production function $Q = 120K^{2/3}L^{2/5}$, where K denotes the capital investment (in units of \$1,000) and L measures the size of the labor force (in worker-hours).
    a. Determine the *marginal productivity of capital*, $\partial Q/\partial K$, and the *marginal productivity of labor*, $\partial Q/\partial L$.
    b. Determine the signs of the second-order partial derivatives $\partial^2 Q/\partial L^2$ and $\partial^2 Q/\partial K^2$, and give an economic interpretation.
49. The temperature at a point (x, y) on a given metal plate in the xy-plane is determined according to the formula $T(x, y) = x^3 + 2xy^2 + y$ degrees. Compute the rate at which the temperature changes with distance if we start at $(2, 1)$ and move
    a. parallel to the vector $\mathbf{j}$
    b. parallel to the vector $\mathbf{i}$
50. In physics, the *wave equation* is

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

and the *heat equation* is

$$\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2}$$

In each of the following cases, determine whether z satisfies the wave equation, the heat equation, or neither.

a. $z = e^{-t}\left(\sin \dfrac{x}{c} + \cos \dfrac{x}{c}\right)$ b. $z = \sin 3ct \sin 3x$
c. $z = \sin 5ct \cos 5x$