Properties of Rings

Theorem: For all a, b in a ring R

(i) $a0 = 0a = 0$.

(ii) $a(-b) = (-a)b = -(ab)$

(iii) $(-a)(-b) = ab$

Pf: (i)

Since R is a ring so it is an additive abelian group with additive identity 0.

For $a \in R$

$0 + a0 = a0$

$\Rightarrow 0 + a0 = a(0+0)$

$\Rightarrow 0 + a0 = a0 + a0$ [$\because$ in a ring Left distributive law holds]

$\Rightarrow 0 = a0$ [by right cancellation law]

$\therefore a0 = 0$.

> **Cancellation laws.**
> Let $(G, *)$ be a group then for all $a, b, c \in G$
> $a * b = a * c \Rightarrow b = c$
> and $b * a = c * a \Rightarrow b = c$

Similarly, ~~0a = 0~~.

$0 + 0a = 0a = (0+0)a$

$\Rightarrow 0 + 0a = 0a + 0a$

$\Rightarrow 0 = 0a$.

(11) for $a, b \in \mathbb{R}$

$$-(ab) + ab = 0 \quad \text{--- (I)}$$

and $a(-b) + ab = a(-b+b)$

$$\Rightarrow a(-b) + ab = a0 = 0 \quad \text{--- (II)}$$

From (I) & (II)

$$-(ab) + ab = a(-b) + ab$$

$$\Rightarrow -(ab) = a(-b) \quad [\text{by cancellation law}]$$

similarly

$$(-a)b + ab = (-a+a)b$$

$$\Rightarrow (-a)b + ab = 0b = 0 \quad \text{--- (III)}$$

From (I) & (III)

$$-(ab) + ab = (-a)b + ab$$

$$\Rightarrow -(ab) = (-a)b$$

$$\therefore a(-b) = (-a)b = -(ab)$$

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(iii)

$(-a)(-b) = -[a(-b)]$ by part (ii)

$= -(-ab)$ by part (ii)

$= ab.$

<u>Theorem</u>: For a, b, c in a ring R

$a(b-c) = ab - ac$ and

$(b-c)a = ba - ca.$

<u>Pf</u> For a, b, c in ~~R~~ R

$a(b+c) = ab + ac$ [by distributive law]

Taking $-c$ in place of c, we get

$a[b+(-c)] = ab + a(-c)$

$\Rightarrow a(b-c) = ab - ac$ [$\because a(-c) = -ac$]

Similarly, $(b-c)a = ba - ca$.

Theorem: If R is a ring with unity element 1, then for all $a \in R$

(i) $(-1)a = -a$, (ii) $(-1)(-1) = 1$.

Pf. We know, for all $a, b \in R$

$$(-a)b = -ab.$$

Taking $a = 1$, $b = a$

$$(-1)a = -1 \cdot a$$

$$\Rightarrow (-1)a = -a$$

Also, we know for all $a, b \in R$

$$(-a)(-b) = ab$$

Taking $a = b = 1$, we get

$$(-1)(-1) = 1 \cdot 1 = 1.$$