* Backward difference operator:

The operator ∇ defined by $\nabla f(x) = f(x) - f(x-h)$ is called the backward difference operator.

The second backward difference of $f(x)$ denoted by $\nabla^2 f(x)$ is the backward difference of the first backward difference $\nabla f(x)$.

i.e $\nabla^2 f(x) = \nabla \{\nabla f(x)\}$

$= \nabla \{ f(x) - f(x-h) \}$

$= \nabla f(x) - \nabla f(x-h)$

$= \{ f(x) - f(x-h) \} - \{ f(x-h) - f(x-2h) \}$

$\Rightarrow \nabla^2 f(x) = f(x) - 2f(x-h) + f(x-2h)$

and so on.

In general the nth backward difference of $f(x)$ is given by $\nabla^n f(x) = \nabla^{n-1} [\nabla f(x)]$

$= \nabla^{n-1} \{ f(x) - f(x-h) \}$

$\Rightarrow \nabla^n f(x) = \nabla^{n-1} f(x) - \nabla^{n-1} f(x-h)$

* Central difference operator δ:

The operator δ defined by

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

is called the central difference operator.

* The shifting operator E :

The operator E defined by $Ef(x) = f(x+h)$, where h is the interval of differencing, is called the shifting operator. By applying the operator E twice, we have

$$E^2 f(x) = E\{Ef(x)\} = Ef(x+h) = f(x+2h),$$

similarly,

$$E^3 f(x) = E\{E^2 f(x)\} = Ef(x+2h) = f(x+3h)$$

and so on.

In general, $E^n f(x) = f(x+nh)$

Also we define E^{-1} by

$$E^{-1} f(x) = f(x-h) \quad \text{so that}$$

$$E^{-n} f(x) = f(x-nh)$$

* Fundamental thm of the difference calculus :

If $f(x)$ be a polynomial of nth degree in x, then the nth difference of $f(x)$ is constant and $\Delta^{n+1} f(x) = 0$

Proof: Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where n is a positive integer and $a_0, a_1, \ldots, a_n$ are constants. Then by definition of Δ, we have

$$\Delta f(x) = f(x+h) - f(x)$$

$$= [a_0 + a_1(x+h) + a_2(x+h)^2 + \cdots + a_n(x+h)^n]$$

$$- [a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n]$$

$$= a_1 h + a_2[(x+h)^2 - x^2] + \cdots + a_n[(x+h)^n - x^n]$$

$$= a_1 h + a_2[{}^2C_1 x h + h^2] + \cdots + a_n[{}^nC_1 x^{n-1} h + {}^nC_2 x^{n-2} h^2 + \cdots + {}^nC_n h^n]$$

$$= b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-2} x^{n-2} + nh\, a_n x^{n-1}$$

where $b_0, b_1, b_2, \ldots, b_{n-2}$ are constant co-efficients.

Thus the first difference of a polynomial of degree n is again a polynomial of degree $n-1$.

Repeating the operation we will observe that $\Delta\Delta f(x)$ i.e $\Delta^2 f(x)$ is a polynomial of degree $(n-2)$ with last term as $n(n-1)h^2 a_n x^{n-2}$.

Continuing this process n times we will get a polynomial of degree $(n-n)$ i.e, of degree zero with the last and only term as $n(n-1)(n-2)\ldots 1 \cdot h^n a_n x^{n-n}$

$$= \underline{n}\, h^n a_n$$

Thus $\Delta^n f(x) = \underline{n}\, h^n a_n$, which is a constant and hence $\Delta^{n+1} f(x) = 0$ and so all higher differences of order $> n$ are also zero.

Note:- The converse of the above theorem is also true i.e. if the nth difference of a function is constant then it may be expressed as a polynomial of degree n.

* Properties of the operators E and Δ.

(i) operators E and Δ are distributive:

Consider a fn $u(x)$ which is the sum of the fns $f(x)$, $g(x)$, $\ldots$ $P(x)$, $\ldots$.

i.e. $u(x) = f(x) + g(x) + P(x) + \ldots$

Then, $Eu(x) = f(x+h) + g(x+h) + P(x+h) + \ldots$

$\Rightarrow Eu(x) = Ef(x) + Eg(x) + EP(x) + \ldots$ [The interval of differencing is taken as h]

i.e. E is distributive

Again,

$$\Delta u(x) = \Delta \{ f(x) + g(x) + P(x) + \cdots \}$$

$$= [f(x+h) + g(x+h) + P(x+h) + \cdots] - [f(x) + g(x) + P(x) + \cdots]$$

$$= \{ f(x+h) - f(x) \} + \{ g(x+h) - g(x) \} + \{ P(x+h) - P(x) \} + \cdots$$

$$= \Delta f(x) + \Delta g(x) + \Delta P(x) + \cdots$$

which shows that Δ is also distributive.

(ii) E and Δ are commutative with regard to constants.

i.e. $E[c f(x)] = c E f(x)$

and $\Delta[c f(x)] = c \Delta f(x)$, where c is constant

We have,

$$E[c f(x)] = c f(x+h), \text{ the interval of differencing being } h.$$
$$= c E f(x)$$

Again, $\Delta[c f(x)] = c f(x+h) - c f(x)$

$$= c \{ f(x+h) - f(x) \}$$
$$= c \Delta f(x)$$

(iii) E and Δ obey the law of indices

i.e. $E^m E^n f(x) = E^{m+n} f(x)$

and $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$

We have,

$$E^m E^n f(x)$$

$$= E^m f(x+nh)$$

$$= f(x+mh+nh)$$

$$= f[x+(m+n)h]$$

$$= E^{(m+n)} f(x)$$

Similarly, we can show that

$$\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$$