Some Other forms of C.L.T.:-

C.L.T. can also be stated as (1) If $X_1, X_2, \ldots, X_n$ are i.i.d. with mean μ_1, variance σ_1^2 and $S_n = X_1 + X_2 + \ldots + X_n$, then for $-\infty < a < b < \infty$

$$\lim_{n\to\infty} P\left[a \le \frac{S_n - n\mu_1}{\sigma_1 \sqrt{n_1}} \le b\right] = \Phi(b) - \Phi(a)$$

$$= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

(2) $$\lim_{n\to\infty} P\left[a \le \frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \le b\right] = \Phi(b) - \Phi(a)$$

or (3) $$\lim_{n\to\infty} P\left[a \le \frac{\bar{X}_n - E(\bar{X}_n)}{\sqrt{V(\bar{X}_n)}} \le b\right] = \Phi(b) - \Phi(a)$$

Relation between CLT & WLLN:-

① Both CLT & WLLN hold for a sequence $\{X_n\}$ of i.i.d. random variables with finite mean μ, variance σ^2.

② For sequence $\{X_n\}$ of independent and uniformly bounded r.v.'s, WLLN holds and CLT holds in this case provided $B_n = Var(X_1 + \ldots + X_n) = \sigma_1^2 + \ldots + \sigma_n^2 \to \infty$ as $n \to \infty$.

③ For the sequence $\{X_n\}$ of independent r.v.'s, CLT may hold but WLLN may not hold.

④ For i.i.d. random variables, CLT is stronger than WLLN. However, WLLN does not require the existence of variance.

Ex:- Let $X_1, X_2, \ldots, X_n$ be i.i.d. Poisson variate with parameter λ. Use CLT to estimate $P[120 \le S_n \le 160]$ where $S_n = X_1 + \ldots + X_n$; $\lambda = 2$, $n = 75$.

Solⁿ:- Given $X_1, X_2, \ldots, X_n$ are i.i.d. random variables which follows Poisson distⁿ with parameter λ, so we have

$$E(X_i) = \lambda, \quad V(X_i) = \lambda \quad (i = 1, 2, \ldots, n)$$

and $E(S_n) = E(X_1 + \ldots + X_n) = \sum_{i=1}^{n} E(X_i) = n.\lambda$

and $V(S_n) = V(X_1 + \ldots + X_n) = \sum_{i=1}^{n} V(X_i) = n\lambda$

Hence by Lindeberg-Levy theorem (for large n),

S_n follows asymptotically normal with mean $= \mu = n\lambda$

$= 75 \times 2$

$= 150$

and variance $= \sigma^2 = n\lambda = 75 \times 2 = 150$.

$$\therefore P(120 \le S_n \le 160) = P\left[\frac{120-150}{\sqrt{150}} \le Z \le \frac{160-150}{\sqrt{150}}\right]$$

(Converting S_n to Z by $\frac{S_n - \mu}{\sqrt{\sigma^2}}$)

$$= P\left[\frac{-30}{12.25} \le Z \le \frac{10}{12.25}\right]$$

$$= P[-2.45 \le Z \le 0.82] \longrightarrow P(a \le Z \le b) = \Phi(b) - \Phi(a)$$

$$= \Phi(0.82) - \Phi(-2.45)$$

$$= \Phi(0.82) - [1 - \Phi(2.45)]$$

$$= [0.5 + 0.2939] - 1 + [0.5 + 0.4929]$$ (using Normal Prob^y table)

$$= 0.7939 - 1 + 0.9929$$

$$= 0.7868$$ //