Differential eqns and Mathematical Models:

The following three examples illustrate the Process of translating scientific laws and Principles into differential eqns. In each of these examples the independent variable is time t, but we will see numerous examples in which some quantity other than time is the independent variable.

Example 1:

Newton's law of cooling may be stated as "The time rate of change (the rate of change wrt time t) of the temperature $T(t)$ of a body is Proportional to the difference between T and the temperature A of the surrounding medium.

i.e. $$\frac{dT}{dt} = -k(T-A) \quad \text{---}(1)$$

where k is a positive constant. Observe that if $T > A$, then $\frac{dT}{dt} < 0$, so the temperature is a decreasing function of t and the body is cooling. But if $T < A$, then $\frac{dT}{dt} > 0$, so that T is increasing.

Thus the physical law is translated into a differential eqn. If we are given the values of k and A, we should be able to find an explicit formula for $T(t)$, and then with the aid of this formula, we can predict the future temperature of the body.

Example 2:

Torricelli's law implies that the time rate of change of the volume V of water in a draining tank is proportional to the square root of the depth y of water in the tank.

i.e.,
$$\frac{dV}{dt} = -K\sqrt{y} \quad \text{----} (2)$$

where K is a constant. If the tank is a cylinder with vertical sides and cross-sectional area A,

then $V = Ay$, so $\frac{dV}{dt} = A\frac{dy}{dt}$.

$$\therefore \frac{dy}{dt} = -\frac{K}{A}\sqrt{y}$$

$$\Rightarrow \frac{dy}{dt} = -h\sqrt{y} \quad \text{----} (3)$$

where $h = \frac{K}{A}$ is a constant.

Example 3:

The time rate of change of a population $P(t)$ with constant birth- and death rates is, in many simple cases, proportional to the size of the population. That is

$$\frac{dP}{dt} = KP \quad \text{----} (4)$$

where K is the constant of proportionality.

Each function of the form $P(t) = ce^{Kt}$ --- (5)

is a solution of the differential equation

$$\frac{dP}{dt} = KP$$

$$\left[\because \; P'(t) = cKe^{Kt} = KP(t), \text{ for all real numbers } t\right]$$

Thus, even if the value of the constant K is known, the differential eq^n $\frac{dP}{dt} = KP$ has infinitely many different solutions of the form $P(t) = ce^{Kt}$, one for each choice of the arbitrary constant c. This is typical of differential eq^ns.