Central Limit Theorem :-

If X_i $(i=1,2,\dots,n)$ be independent random variables such that $E(X_i)=\mu_i$ and $V(X_i)=\sigma_i^2$, then under certain very general conditions, asymptotically the random variable $S_n = X_1+X_2+\dots+X_n$ is normal with mean μ and standard deviation σ,

where $\mu = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$.

This theorem was stated by Laplace in 1812 and later its general conditions was given by Liapounoff in 1901. Some particular cases of this general C.L.T. are given below :-

(I) De-Moivre's theorem :-

A particular case of C.L.T. is De-Moivre's theorem which was given in 1733 and is stated below :-

$$\text{If } X_i = \begin{cases} 1, & \text{with Prob}^y\ p \\ 0, & \text{with Prob } q \end{cases}$$

then, the distn of the random variables

$S_n = X_1 + X_2 + \dots + X_n$, where X_i's are independent,

is asymptotically normal as $n \to \infty$.

Some Remarks :- ① From this theorem, we can say that Binomial distn tends to normal distn as $n \to \infty$.

② Convergence in distn :- Let $\{X_n\}$ be a sequence of random variables and $\{F_n\}$ be their respective distn function. Then we say that X_n converges in distn to X, if there exists a r.v. X with distn function F such that as $n \to \infty$, $F_n(x) \to F(x)$, we can write :-

$$X_n \xrightarrow{d} X \quad \text{as } n \to \infty.$$

③ Here $X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{L} X$, i.e. X_n probly convergent to X implies that X_n converges in law to X. But the converse may not be true, i.e.

$X_n \xrightarrow{L} X$ does not always imply that $X_n \xrightarrow{p} X$.

(II) Lindeberg - Levy Theorem :- This is another form of C.L.T. developed by Lindeberg & Levy which is used for independent and identically distributed (i.i.d.) r.v.'s. This is defined below :-

If $X_1, X_2, \dots, X_n$ are independently & identically distributed random variables with

$$E(X_i) = \mu_1, \quad V(X_i) = \sigma_1^2 \quad (i = 1, 2, \dots, n)$$

then, the sum $S_n = X_1 + X_2 + \dots + X_n$ is asymptotically normal with mean $\mu = n\mu_1$ and variance $\sigma^2 = n\sigma_1^2$.

Here we have to make the following assumptions:-

(i) The variables are independent and identically distributed

(ii) $E(X_i^2)$ exists for all $i = 1, 2, 3, \ldots$

Applications of C.L.T.:- ⓐ If $X_1, X_2, \ldots, X_n \ldots$ are i.i.d. random variables which follows $B(r, p)$, then $S_n = X_1 + X_2 + \cdots + X_n$ will have $E(S_n) = \sum E(X_i) = \sum_{i=1}^{n} rp = nrp$

and $V(S_n) = \sum V(X_i) = \sum_{i=1}^{n} rpq = nrpq$

Which gives another form of C.L.T. as follows:-

$$\lim_{n\to\infty} P\left\{a \le \frac{S_n - nrp}{\sqrt{nrpq}} \le b\right\} = \Phi(b) - \Phi(a),$$

$0 < p < 1,\ p + q = 1.$

ⓑ If Y_n is binomial variate with parameters n, p then

$$\lim_{n\to\infty} P\left[a \le \frac{Y_n - np}{\sqrt{npq}} \le b\right] = \Phi(b) - \Phi(a),\ 0 < p < 1.$$

ⓒ If Y_n is Poisson variate with parameter n, then

$$\lim_{n\to\infty} P\left[a \le \frac{Y_n - n}{\sqrt{n}} \le b\right] = \Phi(b) - \Phi(a),$$ as we can write,

$$\lim_{n\to\infty} P(Y_n \le n) = \frac{1}{2},$$ i.e.

$$\sum_{K=0}^{n} \frac{e^{-n} n^K}{\lfloor K} = \frac{1}{2} \text{ as } n \to \infty.$$