Ex:- Show that the set of all continuous real valued function is a commutative ring with unity under the operation of pointwise addition and multiplication

Soln: Let $F = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous on } \mathbb{R} \}$

for $f, g \in F$ we define their sum and product as follows:

$$(f+g)(x) = f(x) + g(x), \quad \forall x \in \mathbb{R}$$

$$(f.g)(x) = f(x)g(x), \quad \forall x \in \mathbb{R}.$$

F is closed with respect to addition and multiplication:

since $f(x), g(x) \in \mathbb{R}$ so $f(x)+g(x) \in \mathbb{R}$ and $f(x).g(x) \in \mathbb{R}$. ~~Hence f+g is also~~ so $f+g$ and $f.g$ are real valued function. Also since f and g are continuous on $\mathbb{R}$ so $f+g$ and fg are continuous on $\mathbb{R}$. Hence $f+g \in F$ and $f.g \in F$.

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F is commutative w.r.t addition

for $f, g \in F$,

$$(f+g)(x) = f(x)+g(x), \quad \forall x \in \mathbb{R}$$
$$= g(x)+f(x), \quad \forall x \in \mathbb{R} \quad [\because f(x), g(x) \in \mathbb{R}]$$
$$= (g+f)(x)$$

$$\Rightarrow f+g = g+f$$

f is associative w.r.t. addition

for $f, g, h \in F$

$$[(f+g)+h](x) = (f+g)(x)+h(x), \forall x \in \mathbb{R}$$
$$= (f(x)+g(x))+h(x)$$
$$= f(x)+(g(x)+h(x))$$
$$= f(x)+(g+h)(x)$$
$$= [f+(g+h)](x)$$

$$\Rightarrow (f+g)+h = f+(g+h)$$

Existence of zero element:

Consider the function $W: \mathbb{R} \to \mathbb{R}$ s.t.
$W(x) = 0, \forall x \in \mathbb{R}$.

Then clearly $W \in F$

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for any $f \in F$

$(f+\omega)(x) = f(x) + \omega(x) = f(x) + 0 = f(x)$

$\Rightarrow f + \omega = f$

similarly $\omega + f = f$

$\therefore \omega$ is the ~~identity~~ zero element of F.

Existence of inverse: The additive inverse of $f \in F$ is a map $f' : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = -f(x)$, $\forall x \in \mathbb{R}$ because

$(f+f')(x) = f(x) + f'(x) = f(x) - f(x)$

$= 0 = \omega(x)$

$\Rightarrow f + f' = \omega$.

similarly $f' + f = \omega$.

F is multiplicative w.r.t. multiplication.

for $f, g, h \in F$

$[f.(g.h)](x) = f(x)(g.h)(x)$

$= f(x)\,(g(x)\,h(x))$

$= (f(x)\,g(x))\,h(x)$ [$\because f(x), g(x), h(x) \in \mathbb{R}$]

$= (f.g)(x)\,h(x) = [(f.g).h](x)$

$\therefore f.(g.h) = (f.g).h$

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f is distributive. For $f, g, h \in F$

$$\begin{aligned}(f.(g+h))(x) &= f(x).((g+h)(x)) \\ &= f(x).(g(x)+h(x)) \\ &= f(x).g(x) + f(x).h(x) \\ &= (f.g)(x) + (f.h)(x) \\ &= (f.g + f.h)(x)\end{aligned}$$

$\therefore f.(g+h) = f.g + f.h$

Similarly $(g+h).f = g.f + h.f$

$\therefore$ F is a ring.

F is Commutative. For $f, g \in F$

$$\begin{aligned}(fg)(x) &= f(x)\,g(x) = g(x)f(x) \\ &= (gf)(x) \quad , \forall x \in \mathbb{R}\end{aligned}$$

$\therefore f.g = g.f$

Finally let $\theta : \mathbb{R} \to \mathbb{R}$ be the mapping defined by $\theta(x) = 1 \ \forall x \in \mathbb{R}$. Then clearly $\theta \in F$

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Also,

$$(0 \cdot f)(x) = 0(x) f(x) = 1 \cdot f(x)$$
$$= f(x)$$

$$\Rightarrow 0 \cdot f = f$$

Similarly $f \cdot 0 = f$

$\therefore$ 0 is unity of the ring F.

Hence F is a commutative ring with unity.

(HW)

Ex The set of all continuous real valued function of a real variable whose graphs pass through the point (1, 0) is a commutative ring without unity under operations of pointwise addition and multiplication.

Sol: Let $F = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and } f(1) = 0 \}$.

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